# Atmospheric Dynamics

## Lecture 14

Sahraei

https://sci.razi.ac.ir/~sahraei

# Synoptic-Scale Motions

## Quasi-geostrophic Analysis

A primary goal of dynamic meteorology is to interpret the observed structure of large-scale atmospheric motions in terms of the physical laws governing the motions.

These laws, which express the conservation of momentum, mass, and energy, completely determine the relationships among the pressure, temperature, and velocity fields.

### THE OBSERVED STRUCTURE OF EXTRATROPICAL CIRCULATIONS

# Scale Analysis in Isobaric Coordinates

The dynamical equations in isobaric coordinates were developed in Section 3.1 and, for reference, are repeated here.

The horizontal momentum equation, the hydrostatic equation, the continuity equation, and the thermodynamic energy equation may be expressed as

$$\frac{d\vec{V}}{dt} + f\mathbf{k}\times\mathbf{V} = -\nabla\Phi$$

$$\frac{\partial\Phi}{\partial p} = -\alpha = -RT/p$$

$$\nabla\cdot\mathbf{V} + \frac{\partial\omega}{\partial p} = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)T - S_p\omega = J/c_p$$

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + (\mathbf{V}\cdot\nabla)_p + \omega\frac{\partial}{\partial p}$$

$$\omega = \frac{dp}{dt}$$

$$S_p \equiv -T\partial\ln\theta/\partial p$$

$$S_p \approx 5\times10^{-4}\ \mathrm{K\,Pa^{-1}}$$

We first separate the horizontal velocity into geostrophic and ageostrophic parts by letting

$$\mathbf{V} = \mathbf{V}_g + \mathbf{V}_a$$

where the geostrophic wind is defined as $\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi$ *

and the ageostrophic wind, $\mathbf{V}_a$, is just the difference between the total horizontal wind and the geostrophic wind.

We have here assumed that the meridional length scale, L, is small compared to the radius of the earth so that the geostrophic wind (*) may be defined using a constant reference latitude value of the Coriolis parameter.

For the systems of interest $|\mathbf{V}_g| \gg |\mathbf{V}_a|$. More precisely, $|\mathbf{V}_a|/|\mathbf{V}_g| \sim O(\mathrm{Ro})$

Thus, in $\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + (\mathbf{V} \cdot \nabla)_p + \omega \frac{\partial}{\partial p}$ **V** can be replaced by **Vg** and the vertical advection, which arises only from the ageostrophic flow, can be neglected.

The rate of changeof momentum following the total motion is then approximately equal to the rate of change of the geostrophic momentum following the geostrophic wind:

$$\frac{d\vec{V}}{dt} \approx \frac{d_g \vec{V}_g}{dt}$$

$$\frac{d_g}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}_g . \nabla = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

This variation can be approximated by expanding the latitudinal dependence of f in a Taylor series about a reference atitude ¢$_0$ and retaining only the first two terms to yield

$$f = f_0 + \beta y$$

$$\beta \equiv (df/dy)_{\phi_0} = 2\Omega \cos\phi_0 / a \text{ and } y = 0 \text{ at } \phi_0.$$

This approximation is usually referred to as the *midlatitude β-plane* approximation.

For synoptic-scale motions, the ratio of the first two terms in the expansion of f has an order of magnitude

$$\frac{\beta L}{f_0} \sim \frac{\cos\phi_0}{\sin\phi_0}\frac{L}{a} \sim O(\mathrm{Ro}) \ll 1$$

From $$\frac{d\vec{V}}{dt} + f\mathbf{k} \times \mathbf{V} = -\nabla\Phi$$

the acceleration following the motion is equal to the difference between the Coriolis force and the pressure gradient force.

Thus, it is not permissible to simply replace the horizontal velocity by its geostrophic value in the Coriolis term. Rather, we use and

$$\mathbf{V} = \mathbf{V}_g + \mathbf{V}_a \qquad \mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi \;* \qquad f = f_0 + \beta y$$

to write

$$\begin{aligned} f\mathbf{k} \times \mathbf{V} + \nabla\Phi &= (f_0 + \beta y)\,\mathbf{k} \times \left(\mathbf{V}_g + \mathbf{V}_a\right) - f_0\mathbf{k} \times \mathbf{V}_g \\ &\approx f_0\mathbf{k} \times \mathbf{V}_a + \beta y\mathbf{k} \times \mathbf{V}_g \end{aligned}$$

where we have used the geostrophic relation (*) to eliminate the pressure gradient force and neglected the ageostrophic wind compared to the geostrophic wind in the term proportional to βy.

The approximate horizontal momentum equation thus has the form

$$\frac{d_g\vec{V}_g}{dt} = -f_0\mathbf{k} \times \mathbf{V}_a - \beta y\mathbf{k} \times \mathbf{V}_g$$

Each term is thus O(Ro) compared to the pressure gradient force, whereas terms neglected are O($Ro^2$) or smaller.

The geostrophic wind defined in $\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi$ (*) is nondivergent. Thus,

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

and the continuity equation $\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0$ can be rewritten as $\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$

which shows that if the geostrophic wind is defined by (*), ω is determined only by the ageostrophic part of the wind field.

In the thermodynamic energy equation $\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) T - S_p \omega = J/c_p$ the horizontal advection can be approximated by its geostrophic value.

$$T_{tot}(x, y, p, t) = T_0(p) + T(x, y, p, t)$$

Now because $|dT_0/dp| \gg |\partial T/\partial p|$, only the basic state portion of the temperature field need be included in the static stability term, and the quasi-geostrophic thermodynamic energy equation may be expressed in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) T - \left(\frac{\sigma p}{R}\right) \omega = \frac{J}{c_p}$$

$$\sigma \equiv -R T_0 p^{-1} d\ln\theta_0 / dp$$

$$\sigma \approx 2.5 \times 10^{-6} \mathrm{m}^2\,\mathrm{Pa}^{-2}\mathrm{s}^{-2}$$

From $\dfrac{\partial \Phi}{\partial p} = -\alpha = -RT/p$ the thermodynamic energy equation can be expressed in terms of the geopotential field. The result is

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) \left(-\frac{\partial \Phi}{\partial p}\right) - \sigma \omega = \frac{\kappa J}{p}$$

# The quasi-geostrophic equations

$$\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \boldsymbol{\nabla}\Phi$$

$$\frac{D_g \mathbf{V}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{V}_a - \beta y \mathbf{k} \times \mathbf{V}_g$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla}\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

# The Quasi-Geostrophic Vorticity Equation

Just as the horizontal momentum can be approximated to O(Ro) by its geostrophic value, the vertical component of vorticity can also be approximated geostrophically.

In Cartesian coordinates the components of $\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \boldsymbol{\nabla}\Phi$ are

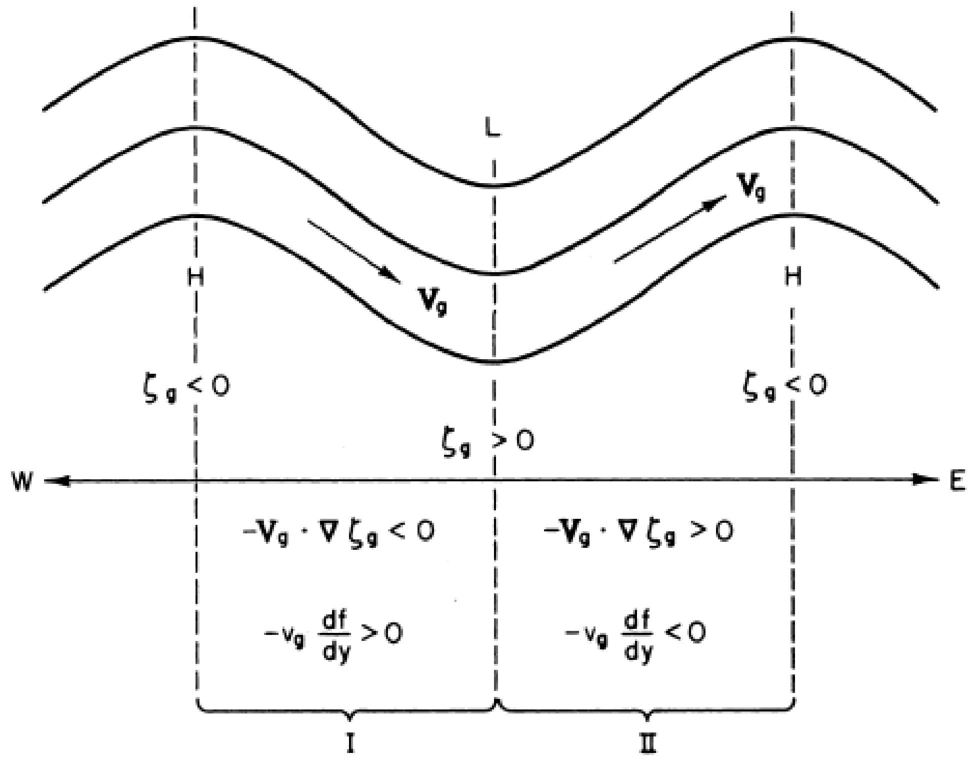
$$f_0 v_g = \frac{\partial \Phi}{\partial x}, \quad f_0 u_g = -\frac{\partial \Phi}{\partial y}$$

$$\zeta_g = \mathbf{k} \bullet \boldsymbol{\nabla} \times \mathbf{V}_g ,$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\boldsymbol{\nabla}^2\Phi$$

This *invertibility* is one reason why vorticity is such a useful forecast diagnostic; if the evolution of the vorticity can be predicted, then inversion of this ewuation yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Since the Laplacian of a function tends to be a maximum where the function itself is a minimum, positive vorticity implies low values of geopotential and vice versa, as illustrated for a simple sinusoidal disturbance in Fig. 6.7.




Schematic 500-hPa geopotential field showing regions of positive and negative advections of relative and planetary vorticity.

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2\Phi$$