Atmospheric Dynamics

Lecture 13

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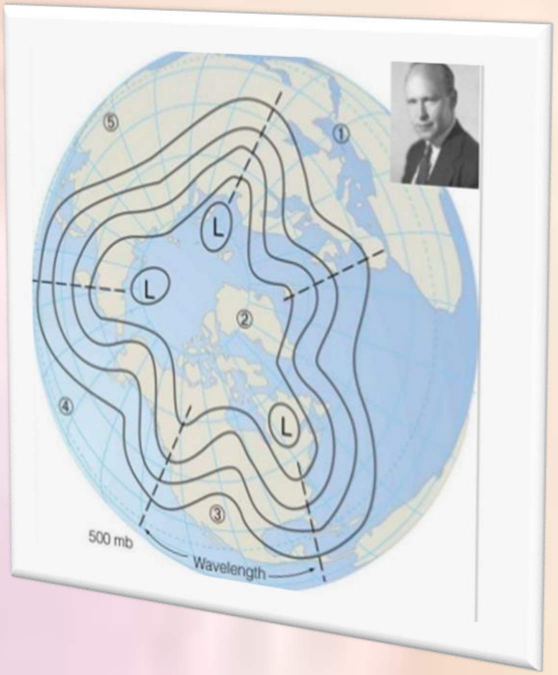
Long-waves in upper-level flow

Rossby waves

4~6 waves

Wavelength
4000~8000 km



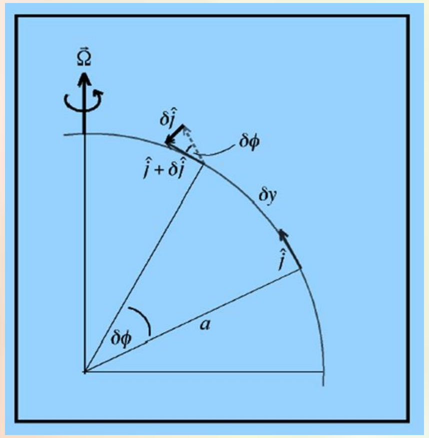
ROSSBY WAVES-ß plane

The variation of the Coriolis parameter with latitude can be approximated by expanding the latitudinal dependence of f in a Taylor series about a reference latitude φ_0 and retaining only the first two terms to yield:

The Coriolis paramter $f = 2\Omega \sin\varphi$ can be expanded using a taylor series around $\varphi = \varphi_o$

$$f = f_0 + \frac{\partial f}{\partial \varphi}\Big|_{\varphi_0} \delta\varphi + \frac{\partial^2 f}{\partial \varphi^2}\Big|_{\varphi_0} \frac{\delta\varphi^2}{2} + \ldots$$

$$f = 2\Omega \sin\varphi_0 + 2\Omega \cos\varphi_0 \delta\varphi + \frac{\partial^2 f}{\partial \varphi^2}\Big|_{\varphi_0} \frac{\delta\varphi^2}{2} + \ldots$$



$$f = f_0 + \beta y$$

$$\beta = \left(\frac{\partial f}{\partial y}\right)_{\varphi_0} = \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial y} = 2\Omega \cos\varphi_0 / a \qquad y = a\delta\varphi \qquad y = 0 \ at \ \varphi_0$$

At the equator, $f_0 = 0$ where upon $f = \beta y$

(a is the radius of earth)

We call this an equatorial beta-plane approximation.

The approximation is usually referred to as the midlatitude β-plane approximation

The Coriolis parameter changes linearly with latitude

The f-plane approximates f as the first term of the Taylor Series, f is taken as a constant.

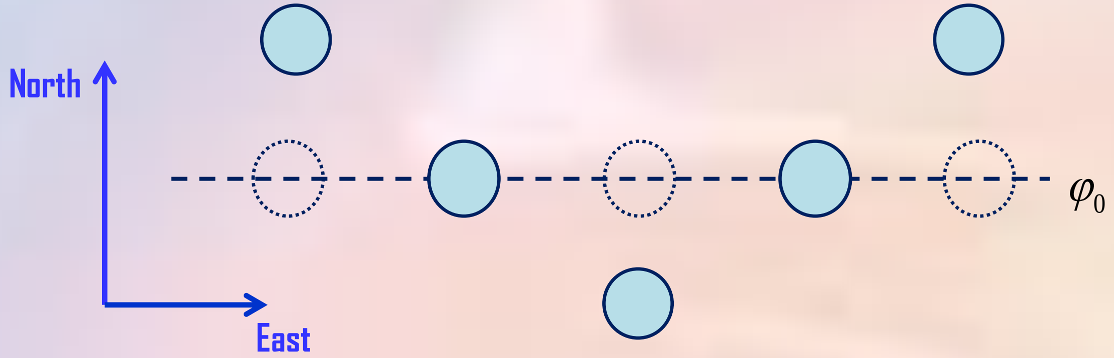
Rossby wave propagation can be understood in a qualitative fashion by considering a closed chain of fluid parcels initially aligned along a circle of latitude.

Planetary Waves are Inertial Waves

A planetary wave in its pure form is a type of inertial wave which owes its existence to the variation of the Coriolis parameter with latitude.

An inertial wave is one in which energy transfer is between the kinetic energy of relative motion and kinetic energy of absolute motion.

Such waves may be studied within the framework of the Cartesian equations described above by making the so-called "beta-plane" or "beta-plane" approximation.



Recall that the absolute vorticity $\eta = \zeta + f$

Assume that $\zeta = 0 \ at \ time \ t_0$

Now suppose that at t_1 , δy is the meridional displacement of a fluid parcel from the original latitude.

Then at t_1 we have $(\zeta + f)_{t_1} = f_{t_0}$

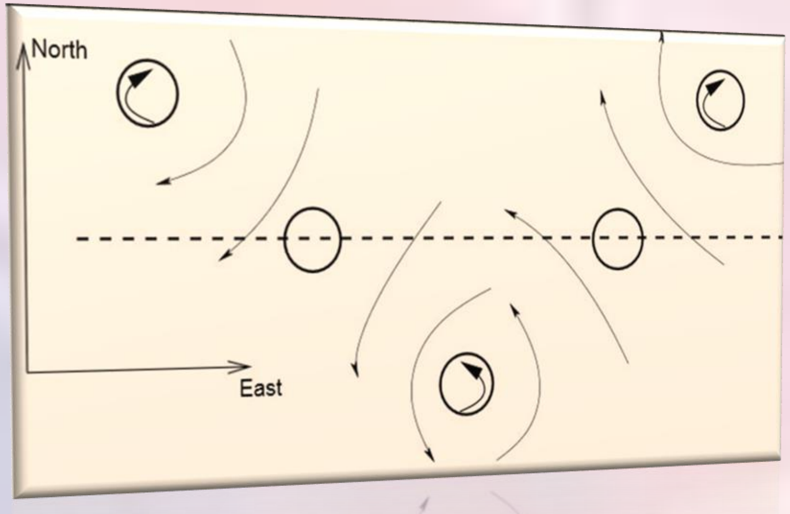
$$\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta \delta y$$

$\beta = df / dy$ is the planetary vorticity gradient at the original latitude

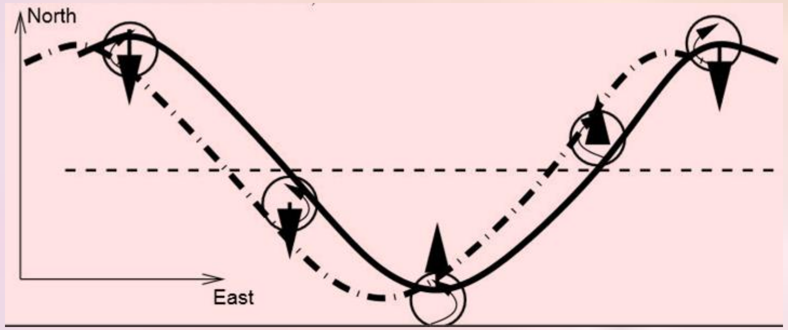
Variation of the Coriolis parameter with latitude, the so called β effect

$$\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta\delta y$$

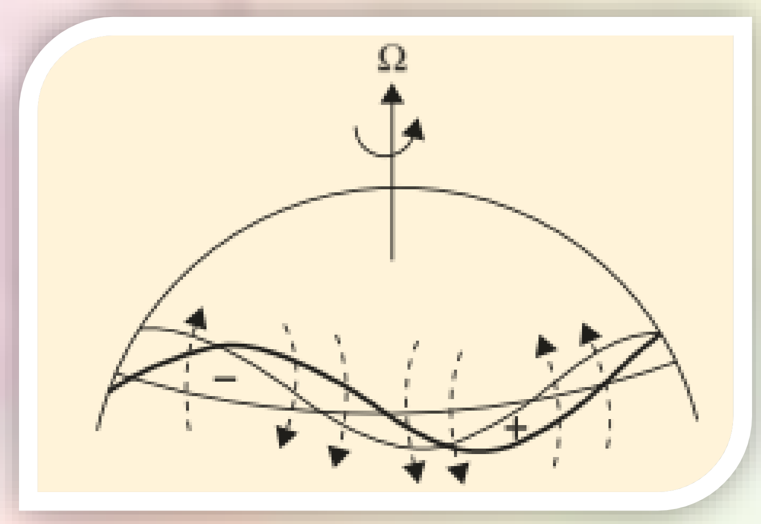
It is evident that if the chain of parcels is subject to a sinusoidal meridional displacement under absolute vorticity conservation, the resulting perturbation vorticity will be positive for a southward displacement and negative for a northward displacement.



This perturbation vorticity field will induce a meridional velocity field, which advects the chain of fluid parcels southward west of the vorticity maximum and northward west of the vorticity minimum, as indicated in Fig.



Perturbation vorticity field and induced velocity field (dashed arrows) for a meridionally displaced chain of fluid parcels.



$\zeta_{t_1} = -\beta\delta y < 0$ $\quad\quad$ $\zeta_{t_1} = -\beta\delta y > 0$

The heavy wavy line shows original perturbation position; the light line shows westward displacement of the pattern due to advection by the induced velocity.

The speed of westward propagation, c, can be computed for this simple example by letting

$$\delta y = a \sin\left[k(x-ct)\right]$$

where a is the maximum northward displacement

$$v = \frac{d(\delta y)}{dt} = -kca\cos\left[k(x-ct)\right]$$

$$\zeta = \frac{\partial v}{\partial x} = -k^2 ca \sin\left[k(x-ct)\right]$$

Substitution for δy *and ξ in*

$$\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta\delta y$$

$$k^2 ca \sin\left[k(x-ct)\right] = -\beta a \sin\left[k(x-ct)\right] \qquad c = -\frac{\beta}{k^2}$$

Thus, the phase speed is westward relative to the mean flow and is inversely proportional to the square of the zonal wave number.

Free Barotropic Rossby Waves

The dispersion relationship for barotropic Rossby waves may be derived formally by finding wave-type solutions of the linearized barotropic vorticity equation.

The barotropic vorticity equation $\frac{d_h(\zeta + f)}{dt} = 0 \quad \rightarrow (\zeta + f) = \text{costant}$

states that the vertical component of absolute vorticity is conserved following the horizontal motion.

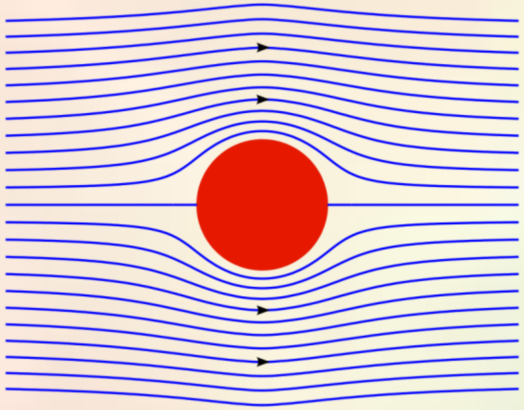
For a midlatitude -plane this equation has the form:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0$$

$$\text{since} \qquad \frac{d_h f}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = v\frac{\partial f}{\partial y} \equiv \beta v$$

We now assume that the motion consists of a constant basic state zonal velocity plus a small horizontal perturbation:

$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad \zeta = \bar{\zeta} + \zeta'$$



The perturbation form: $$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \bar{v}\frac{\partial \zeta'}{\partial y} + \beta v' = 0$$

If we assume $\bar{v} = 0$: $$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \beta v' = 0$$

To solve this PDE, we recall: $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \rightarrow \zeta' = \frac{\partial v'}{\partial x} - \cancel{\frac{\partial u'}{\partial y}} = \frac{\partial v'}{\partial x}$$

We define a perturbation stream function $\psi'(x, y, t)$ according to:

$$\frac{\partial \psi'}{\partial x} = v' \qquad \frac{\partial \psi'}{\partial y} = -u'$$

$$\zeta' = \nabla^2 \psi'$$

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \beta v' = 0$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2 \psi' + \beta\frac{\partial \psi'}{\partial x} = 0 \quad *$$

where as usual we have neglected terms involving the products of perturbation quantities. We seek a solution of the form

$$\psi' = \Psi e^{i(kx+ly-\omega t)} = \Psi e^{i\phi} \qquad \phi = (kx + ly - \omega t)$$

Here k and l are wave numbers in the zonal and meridional directions, respectively

Substituting for ψ' in * $\qquad \nabla^2 \psi' = -\left(k^2 + l^2\right)\psi'$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2 \psi' = -\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(k^2 + l^2\right)\psi' = -\left(k^2 + l^2\right)\left(-i\omega + \bar{u}ik\right)\psi'$$

$$\beta\frac{\partial\psi'}{\partial x}=\beta ik\psi'$$

This gives the dispersion relation

$$-\left(k^2+l^2\right)\left(-\omega+\bar{u}k\right)+\beta k=0$$

$$c_g \equiv \frac{\partial\omega}{\partial k} = U - \frac{\beta(l^2-k^2)}{(k^2+l^2)^2},$$

$$\boxed{\omega=\bar{u}k-\frac{\beta k}{k^2+l^2}}$$

$$\omega = -\beta/k,$$

$$c_g=\frac{\beta}{k^2}$$

$$\boxed{\omega=\bar{u}k-\frac{\beta k}{K^2}}$$

$K^2\equiv k^2+l^2$ is the total horizontal wave number squared.

Phase speed in the x direction is $c_x=\dfrac{\omega}{k}=\bar{u}-\dfrac{\beta}{K^2}$

$c_R=-\dfrac{\beta}{K^2}$ Depend on the variation of the Coriolis parameter with latitude and wavelength

Non-divergent motions

A simple description of the basic dynamics of a pure horizontally nondivergent planetary wave may be given by considering two-dimensional flow on a beta plane:

i.e., in a rectangular coordinate system with x pointing eastwards, y pointing northwards, and with $f = f_0 + \beta y$.

Horizontal nondivergence implies that H is a constant and therefore, in the absence of a body force, the vorticity equation reduces to

$$\frac{d}{dt}(f+\zeta)=0$$

The absolute vorticity of each fluid column remains constant throughout the motion.

Rossby Waves

Planetary Waves are Inertial Waves

- ✓ Rossby waves, also known as *planetary waves* owe their origin to the shape and rotation of the earth
- ✓ Travel from ***east to west***, following latitude
- ✓ Slow moving
  - ✓ Speed varies with latitude slower near the pole, faster near the Equator
  - ✓ On the order of a few cm/s (or a few km/day)

$c_x = phase\ speed$

$\bar{u} = zonal\ flow\ at\ 500\ hPa$

$K = wave\ enumers$

$\beta = \frac{df}{dy}$

Phase speed in the x direction is $c_x = \frac{\omega}{k} = \bar{u} - \frac{\beta}{K^2}$

$c_R = -\frac{\beta}{K^2}$ Depend on the variation of the Coriolis parameter with latitude and wavelength

Coriolis Acceleration: Approximations "Tangent Planes"

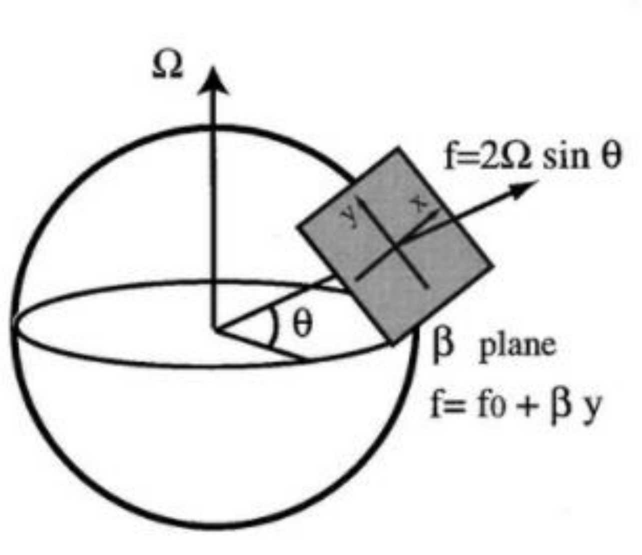
- full latitude dependence: $f = 2\Omega\sin(\theta)$

- *f*-plane approximation: $f = f_o = 2\Omega\sin(\theta_o)$

- *β*-plane approximation: $f = f_o + \beta y$

$$\beta = \frac{2\Omega\cos(\theta_o)}{R}$$

The beta-plane approximation allows us to study of the effects of varying f with latitude without the added complication of working in spherical geometry.



The zonal phase speed relative to the mean wind is

$$c_x - \bar{u} = -\frac{\beta}{K^2} < 0 \qquad c_x = \bar{u} - \frac{\beta L^2}{4\pi^2}$$

when the mean wind vanishes and $l \to 0$

Thus, the Rossby wave zonal phase propagation is always westward relative to the mean zonal flow.

Furthermore, the Rossby wave phase speed depends inversely on the square of the horizontal wave number.

Therefore, Rossby waves are dispersive waves whose phase speeds increase rapidly with increasing wavelength $(c_x \propto 1/K^2)$

$$c_x = \bar{u} - \frac{\beta}{K^2} \qquad c_x = \bar{u} - \frac{\beta L^2}{4\pi^2}$$

$c_x = phase\ speed$

$\bar{u} = zonal\ flow\ at\ 500\ hPa$

$K = wave\ enumers$

$$\beta = \frac{df}{dy}$$

It is noted that the zonal phase speed of Rossby waves is always westward (traveling east to west) relative to mean flow $\bar{u}$, but the zonal group speed of Rossby waves can be eastward or westward depending on wave number.

For a typical midlatitude synoptic-scale disturbance, with similar meridional and zonal scales $(k \approx l)$ and zonal wavelength of order 6000 km, the Rossby wave speed relative to the zonal flow calculated from:

$$c_x - \bar{u} = -\frac{\beta}{2k^2} = \frac{L_x^2}{8\pi^2}\frac{d}{dy}2\Omega\sin\varphi = -\frac{L_x^2}{8\pi^2}\frac{d}{Rd\varphi}2\Omega\sin\varphi$$

$$= -\frac{\Omega L_x^2}{4R\pi^2}\frac{d}{d\varphi}\sin\varphi = -\frac{7\times10^{-5}\times36\times10^{12}}{4\times6.4\times10^6\times10}\cos 45^\circ \approx -8\,ms^{-1}$$

Because the mean zonal wind is generally westerly (to the east) and greater than 8 ms^{-1} .

Synoptic-scale Rossby waves usually move eastward, but at a phase speed relative to the ground at a lower speed.

Relation between stream lines and trajectories in a progressive flow

$$c_x = \bar{u} - \frac{\beta}{K^2} = \bar{u} - \frac{\beta L^2}{4\pi^2}$$

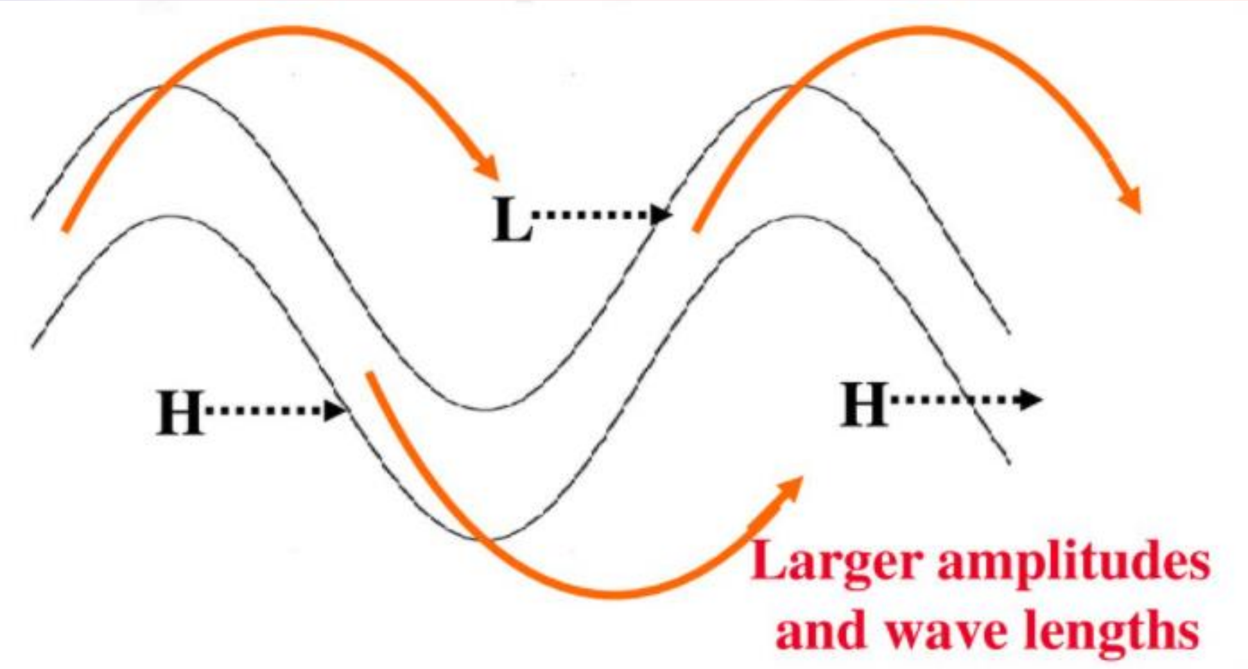
1) $c_x > 0 \quad i.e \quad \bar{u} - \frac{\beta}{K^2} > 0 \quad i.e \quad \bar{u} > \frac{\beta}{K^2}$

$\longrightarrow \bar{u}$

$-c_R \longleftarrow$

$\longrightarrow c_x$

$c_x > 0$ for small L



Relation between stream lines and trajectories in a retrogressive flow

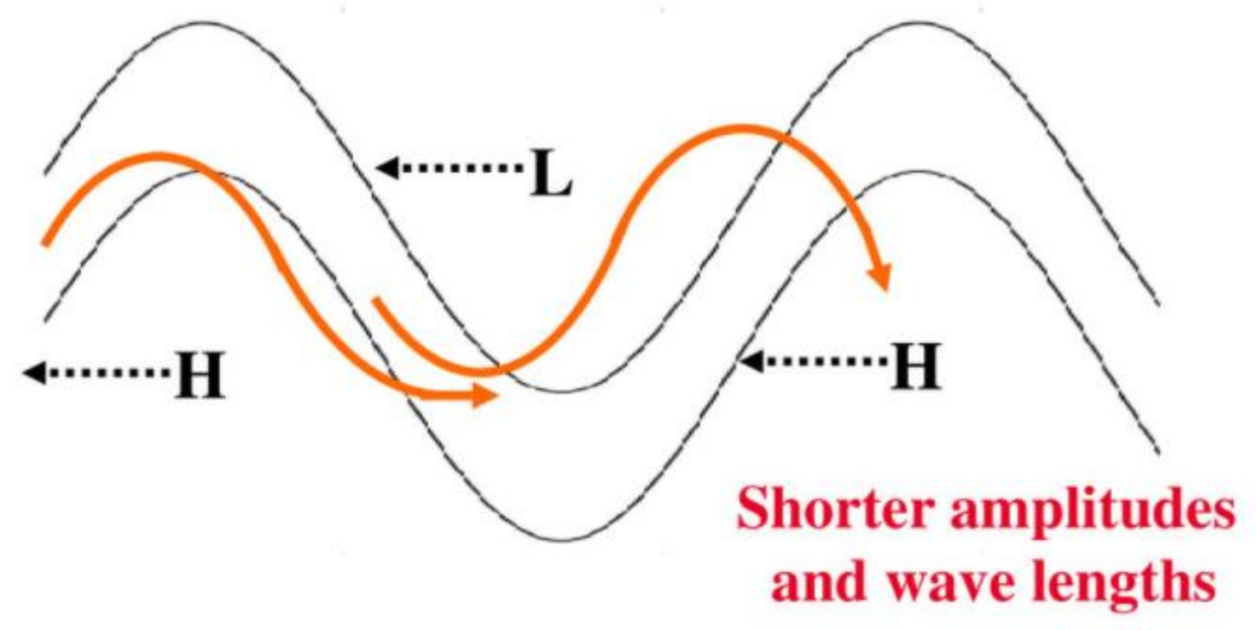
2) $c_x < 0$ i.e $\bar{u} - \frac{\beta}{K^2} < 0$ i.e $\bar{u} < \frac{\beta}{K^2}$

$\xrightarrow{\hspace{2em}} \bar{u}$

$-c_R \xleftarrow{\hspace{2em}}$

$c_x \xleftarrow{\hspace{1em}}$

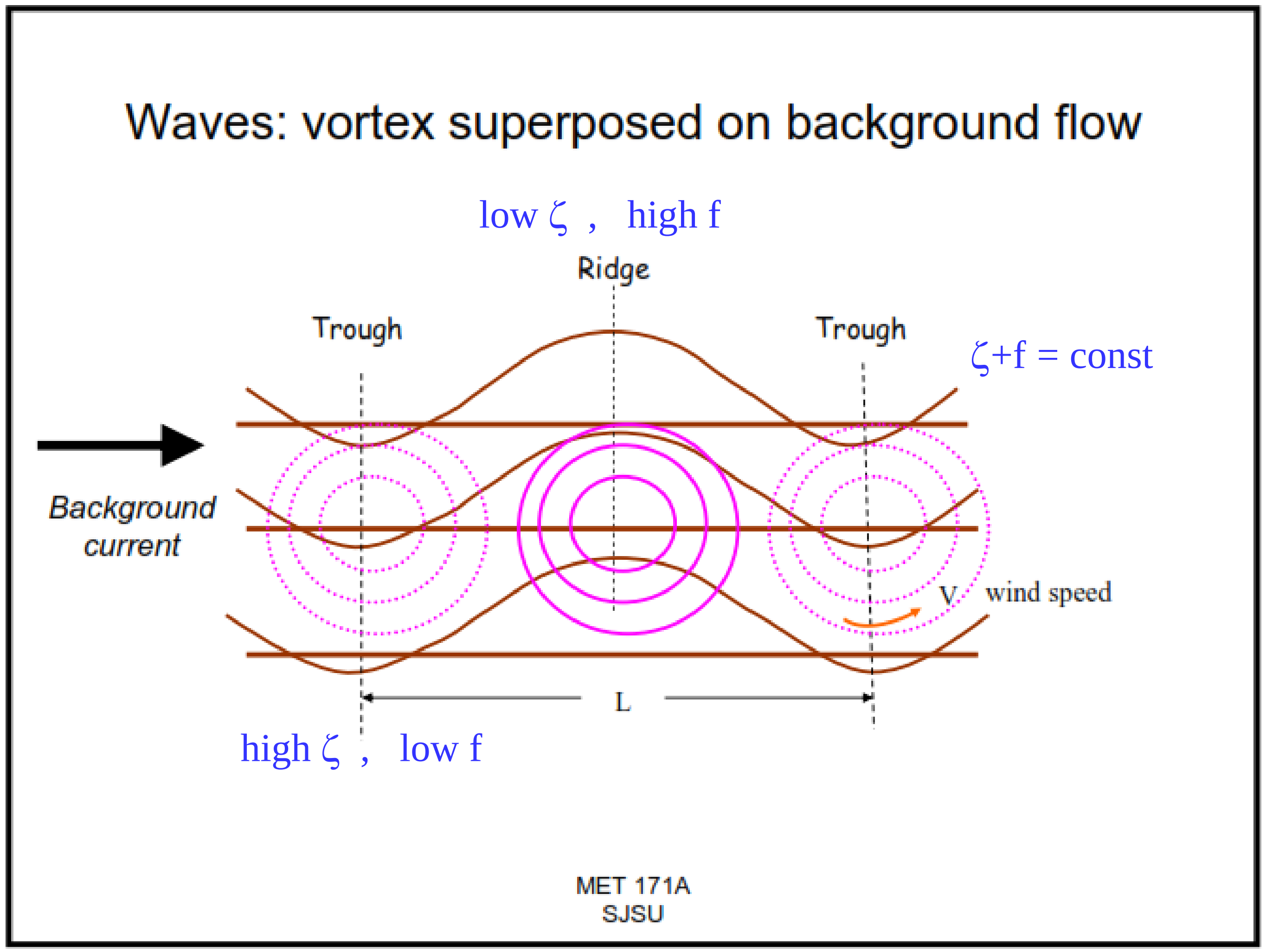
$c_x < 0$ for large L



Note that $\beta > 0$ implies that $c_x < 0$ and hence the waves travel towards the west

A very common misunderstanding:

This is NOT a Rossby wave!



...but a Constant Absolute Vorticity Trajectory!

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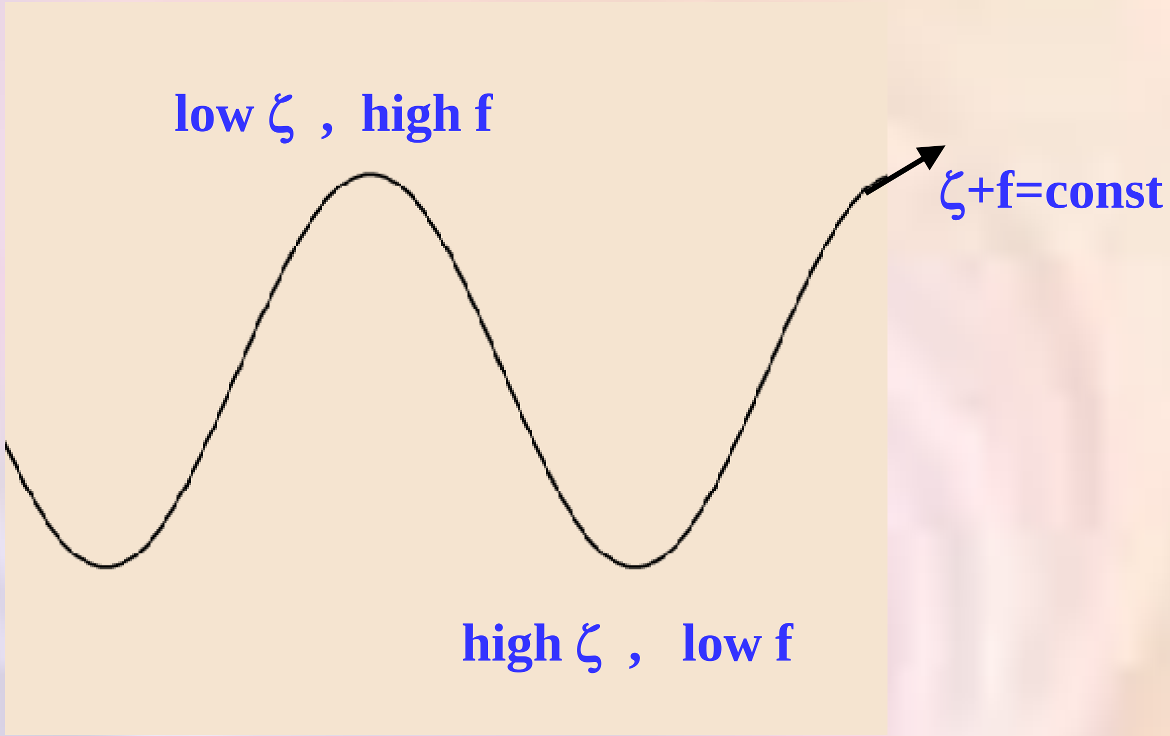
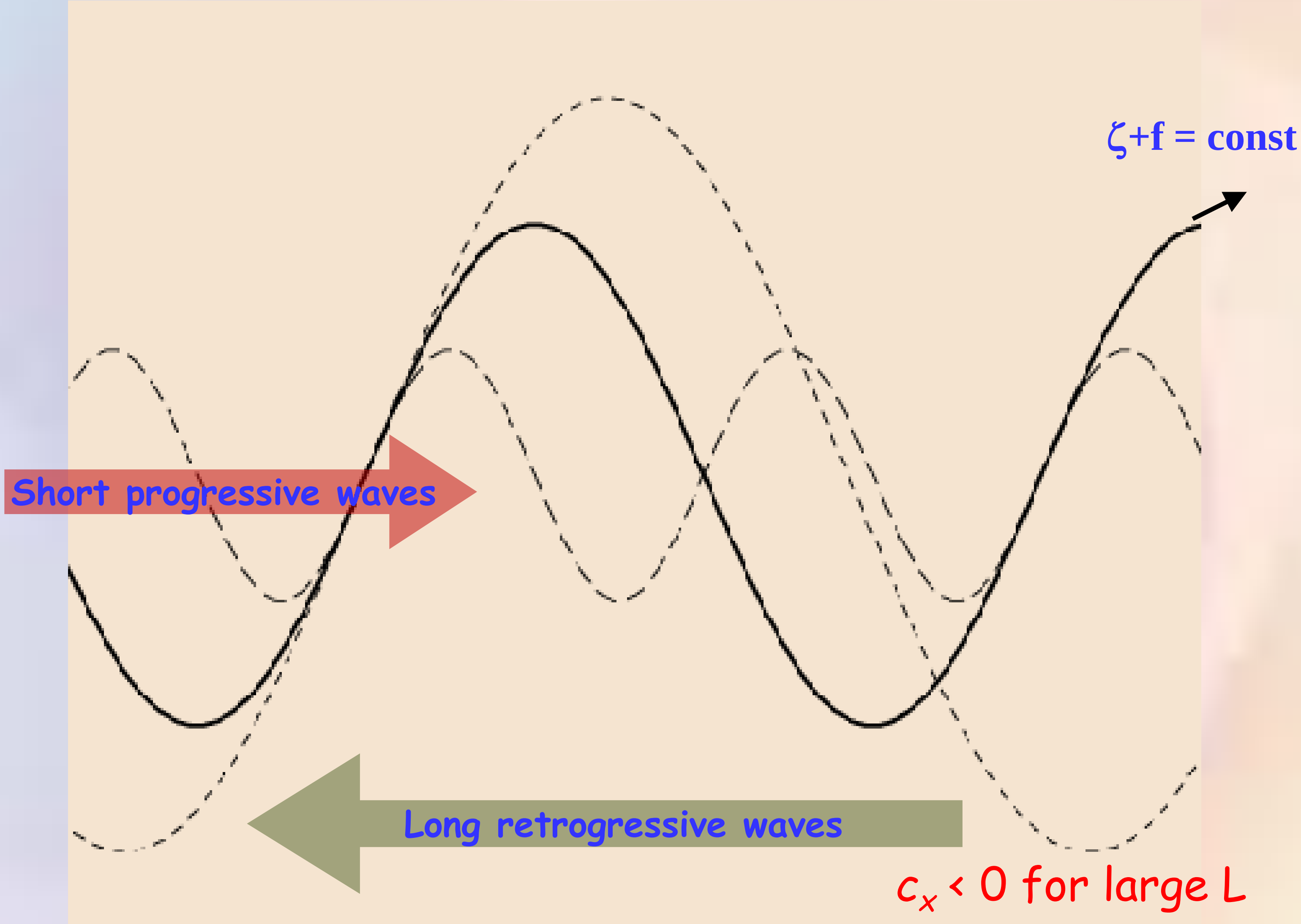


…but a Constant Absolute Vorticity Trajectory!

One and the same CAV trajectory satisfies

two types of streamlines (waves)



Stationary Rossby Wave

$$c_x = \bar{u} - \frac{\beta}{K^2}$$

For longer wavelengths the westward Rossby wave phase speed may be large enough to balance the eastward advection by the mean zonal wind so that the resulting disturbance is stationary relative to Earth's surface.

$$3)\quad c_x = 0 \quad i.e \quad \bar{u} = \frac{\beta}{K^2} \quad \rightarrow K^2 = \frac{\beta}{\bar{u}} \equiv K_s^{\,2} \qquad \lambda_s = 2\pi\sqrt{\frac{\bar{u}}{\beta}}$$

For stationary waves can propagate only in eastward zonal winds ($\bar{u} > 0$) that are not too strong are listed below for β as 1.6 x 10^{-11} m $^{-1}$s $^{-1}$, appropriate to 45 deg. latitude.

Since c_x is a function of k (or λ), the waves are called dispersive.

In this case, the longer waves travel faster than the shorter waves.

$\bar{u}$ (ms^{-1})	20	40	60	80
λ (10^3 km)	7.0	9.0	12.0	14.0