Atmospheric Dynamics

Lecture 11

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Wave Generation

Disturbing force causes waves to form

Wind blowing across ocean surface (most surface ocean waves)

Interface of fluids with different densities

Air – ocean interface - Ocean waves

Air – air interface - Atmospheric waves

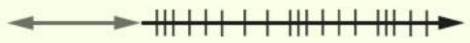
Water – water interface - Internal waves

Internal waves often larger than surface waves

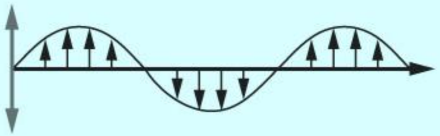
Mass movement into ocean

Splash waves

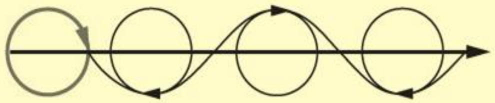
Types of waves



(1) LONGITUDINAL WAVE
Particles (color) move back and forth in direction of energy transmission. These waves transmit energy through all states of matter.

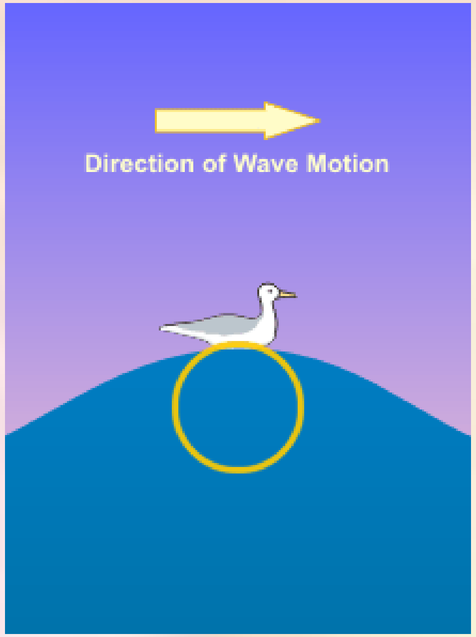


(2) TRANSVERSE WAVE
Particles (color) move back and forth at right angles to direction of energy transmission. These waves transmit energy only through solids.



(3) ORBITAL WAVE
Particles (color) move in orbital path. These waves transmit energy along interface between two fluids of different density (liquids and/or gases).

Wave particles move in a circle



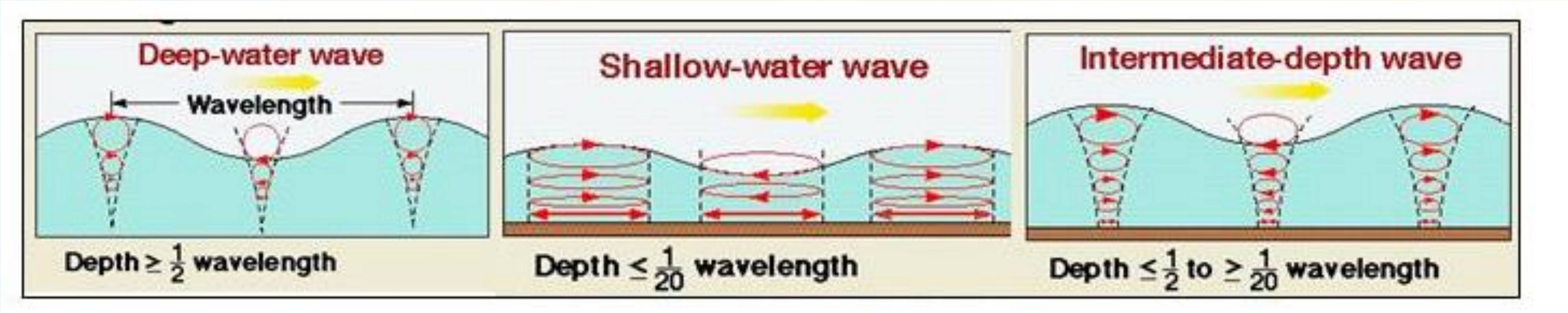
Waves are moving energy

Progressive waves in

(a) deep waters

(b) shallow waters

(c) intermediate waters

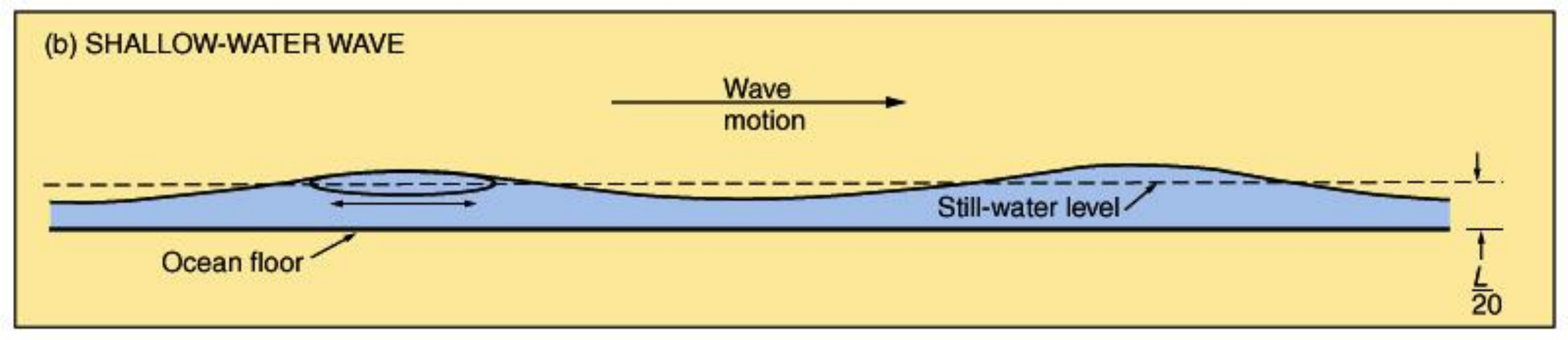


Notice that the orbital motion changes significantly according with the depth and the relationship between depth and wavelength.

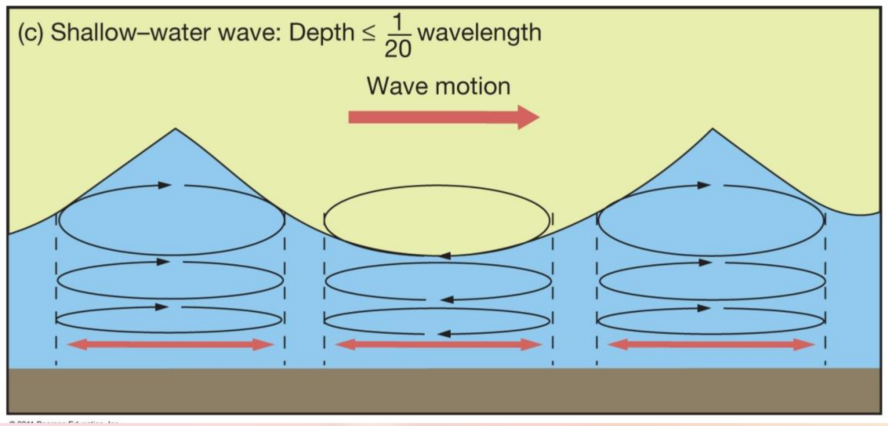
Shallow-water wave

Water depth is less than 1/20 wavelength

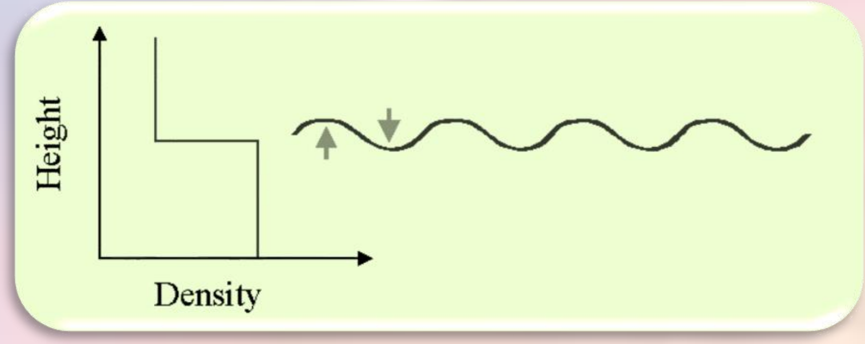
Wave speed (celerity) is proportional to depth of water



Orbital motion is flattened



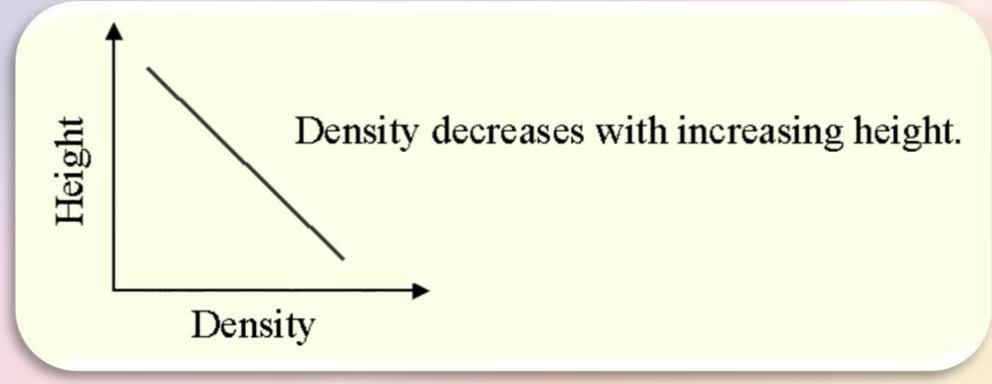
Restoring Force: Gravity



Force is transverse to direction of propagation

Gravity waves are buoyancy waves, the restoring force comes from Archimedes's principle.

They involve vertical displacement of air parcels, along slanted paths



Shallow-Water Wave

The second example of pure wave motion concerns the horizontally propagating oscillations known as shallow water waves.

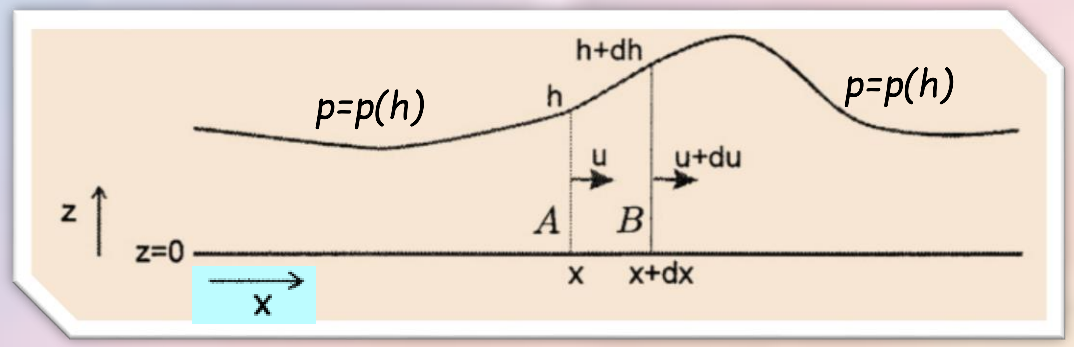
1- Incompressible and homogeneous flow, where ρ_0 is a constant density => no sound waves (simplifies equations)

2- The flow is assumed to be inviscid

3-The water is so shallow that the flow velocity, $V(x, y)$, is constant with depth.

4- Shallow water, Require $\lambda_x \gg h$. Otherwise too deep for hydrostatic assumption.

$$\frac{\partial p}{\partial z} = \rho_0 g$$ hydrostatic approximation



Consider the volume of water bounded by vertical surfaces A and B in Figure. These surfaces are located at x and x+dx respectively.

We will now eliminate the vertical velocity w, thereby reducing the system to three equations for three variables.

z = h(x, y), the height of the free surface at point (x,y) that, the pressure is equal to atmospheric pressure p(h), assumed constant and uniform.

Integrating the hydrostatic equation over the depth of the fluid, *h(x, y)*, gives the pressure between *z and h* below the surface

Shallow water model Matsuno (1966)

$$\int_z^h \frac{\partial p}{\partial z} dz + \int_z^h \rho_0 g dz = 0 \qquad p(z) = p(h) + \rho_0 g(h - z) \ *$$

where p(h) is the pressure at the top of the layer of shallow water due to the layer above, which we take to be a constant.

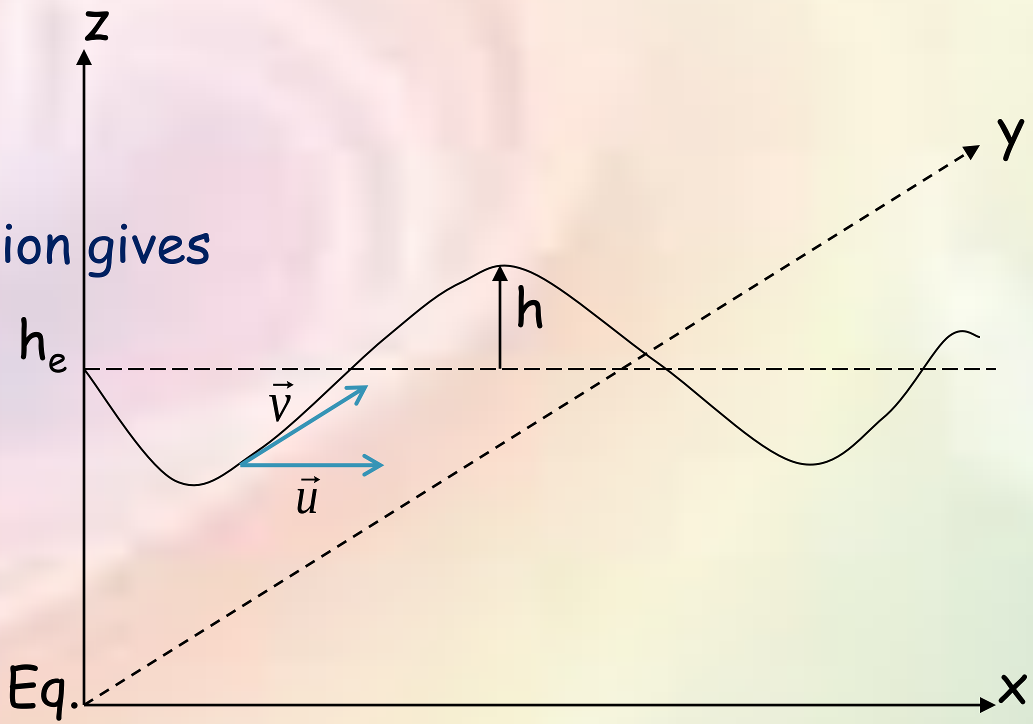
We assume that **V** is initially a function of (x, y) only, and since h is a function of (x, y).

*This equation indicates that **V** will remain two-dimensional for all time.*

Using * to replace pressure in the momentum equation gives

$$\frac{d_h \vec{V}}{dt} = -g\nabla_h h - f\hat{k} \times \vec{V} \quad *'$$

$$-\frac{1}{\rho}\nabla p = -g\nabla_h h$$



Mass conservation for a constant density flow has the simple form

$$\nabla.(u, v, w) = 0$$

Next, we integrate the continuity equation through the full depth of the fluid. Since u and v are constant with z,

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz = \int_0^h \left(\frac{\partial w}{\partial z}\right)dz$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -w(h) + w(0) = -\frac{dh}{dt}$$

Here we have assumed that the bottom boundary is flat, so that the vertical velocity there vanishes: $w(0) = 0$.

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

The shallow water equations consist of

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - fv + g\dfrac{\partial h}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + fu + g\dfrac{\partial h}{\partial y} = 0 \\ \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + v\dfrac{\partial h}{\partial y} + h(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}) = 0 \end{cases}$$

The linearized momentum equation mass continuity equations are:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv'$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu'$$

$$\frac{\partial h'}{\partial t} = -\bar{h}(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y})$$

Primes denote perturbation values, that is, departures from the state of rest.

These equations represent a set of three coupled first-order partial differential equations in the unknown u', v', h'

One approach to solving these equations is to form and solve a single third-order partial differential equation.

This is accomplished by taking $\frac{\partial}{\partial t}$ *of the third equation, which gives*

$$\frac{\partial^2 h'}{\partial t^2} = -\bar{h}\left(\frac{\partial^2 u'}{\partial t \partial x} + \frac{\partial^2 v'}{\partial t \partial y}\right)$$

The terms on the right side of this equation may be replaced using the first two equations of

$$\frac{\partial^2 h'}{\partial t^2} = -\bar{h}\left(g\nabla_h^2 h' - f\zeta'\right)$$

Again take $\frac{\partial}{\partial t}$ *which gives the third-order equation*

$$\frac{\partial^3 h'}{\partial t^3} + \left(f^2 - g\bar{h}\nabla_h^2\right)\frac{\partial h'}{\partial t} = 0 \;\; *$$

where $\partial\zeta'/\partial t$ is replaced using the linearized version of

$$\frac{d_h}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

(noting that f is taken constant here):

$$\frac{\partial\zeta'}{\partial t} = -f(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y})$$

Assuming that the lateral boundaries are periodic and, since the coefficients f *and* $g\bar{h}$ constant, we may assume wave solutions of the form

$$h' = \mathrm{Re}\{Ae^{i(kx+ly-\omega t)}\} \quad *'$$

Using *' in *gives a cubic polynomial for the frequency

$$\omega^3 - \omega[f^2 + g\bar{h}(k^2 + l^2)] = 0$$

This is the dispersion relationship for shallow water waves.

Clearly $\omega = 0$ is a solution, and if $\omega \neq 0$, then

$$\omega^2 = f^2 + g\bar{h}(k^2 + l^2)$$

For readers familiar with linear algebra, we note an alternative solution method.

The solution of the form $h' = \mathrm{Re}\left\{Ae^{i(kx+ly-\omega t)}\right\}$ for h', u', and v', and substituting directly into

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv' \qquad \frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu' \qquad \frac{\partial h'}{\partial t} = -\bar{h}(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y})$$

converts the set of partial differential equations to algebraic equations that may be written as

$$Ax = 0$$

where x is the column vector of the unknowns $\begin{bmatrix} u' \\ v' \\ h' \end{bmatrix}$

$$\begin{bmatrix} -i\omega & -f & ikg \\ f & -iv & ilg \\ \bar{h}k & -\bar{h}l & -\omega \end{bmatrix} \begin{bmatrix} u' \\ v' \\ h' \end{bmatrix}$$

A nontrivial solution to is obtained only if A is not invertible.

This is enforced by setting the determinant of A to zero, which gives

$$\omega^3 - \omega\left[f^2 + g\bar{h}(k^2 + l^2)\right] = 0$$