Geophysical Fluid Dynamics

Lecture 9

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Total Differentiation of a Vector in a Rotating Frame of Reference

Before we can write Newton's second law of motion for a reference frame rotating with the earth, we need to develop a relationship between the total derivative of a vector in an inertial reference frame and the corresponding derivative in a rotating system.

Let $\vec{A}$ be an arbitrary vector with Cartesian components

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$ in an inertial frame of reference, and

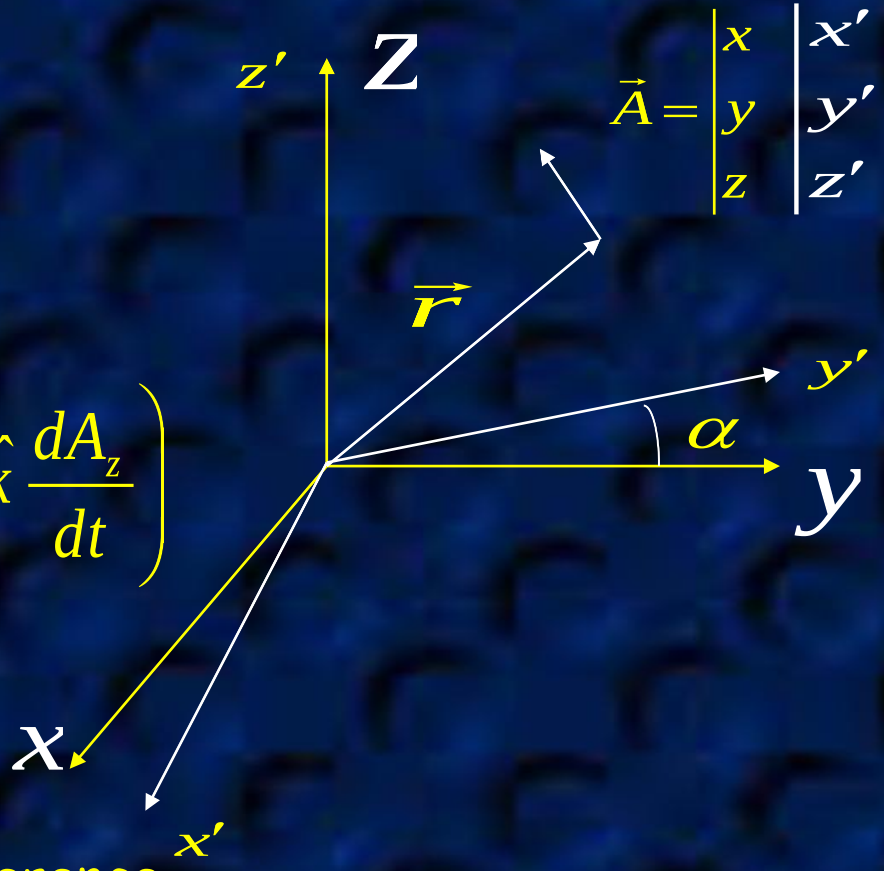
$$\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$$ in a rotating frame of reference

If

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

in an inertial frame of reference, then

$$\frac{d\vec{A}}{dt} = \left(A_x\frac{d\hat{i}}{dt} + \hat{i}\frac{dA_x}{dt}\right) + \left(A_y\frac{d\hat{j}}{dt} + \hat{j}\frac{dA_y}{dt}\right) + \left(A_z\frac{d\hat{k}}{dt} + \hat{k}\frac{dA_z}{dt}\right)$$



Since the coordinate axes are in an inertial frame of reference,

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} \qquad \textbf{(Eq. 1)}$$

If $\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$ in a rotating frame of reference, then

$$\frac{d\vec{A}}{dt} = \left(A'_x\frac{d\hat{i}'}{dt} + \hat{i}'\frac{dA'_x}{dt}\right) + \left(A'_y\frac{d\hat{j}'}{dt} + \hat{j}'\frac{dA'_y}{dt}\right) + + \left(A'_z\frac{d\hat{k}'}{dt} + \hat{k}'\frac{dA'_z}{dt}\right) \qquad \textbf{(Eq. 2)}$$

Because the left hand sides of (Eq. 1) and (Eq. 2) are identical,

$$\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} = \left(A'_x \frac{d\hat{i}'}{dt} + \hat{i}'\frac{dA'_x}{dt}\right) + \left(A'_y \frac{d\hat{j}'}{dt} + \hat{j}'\frac{dA'_y}{dt}\right) + \left(A'_z \frac{d\hat{k}'}{dt} + \hat{k}'\frac{dA'_z}{dt}\right)$$

Regrouping the terms

$$\underbrace{\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x\frac{d\hat{i}'}{dt} + A'_y\frac{d\hat{j}'}{dt} + A'_z\frac{d\hat{k}'}{dt}}_{\text{effects of rotation}}$$

To interpret $\frac{d\hat{i}'}{dt}, \frac{d\hat{j}'}{dt}, \frac{d\hat{k}'}{dt}$

think of each unit vector as a position vector

linear velocity = angular velocity x position vector

$$\vec{V} = \vec{\Omega} \times \vec{r}$$

Because $\vec{V} = \frac{d\vec{r}}{dt}$, $\frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r}$

Thus $\frac{d\hat{i}'}{dt} = \vec{\Omega} \times \hat{i}'$, $\frac{d\hat{j}'}{dt} = \vec{\Omega} \times \hat{j}'$, $\frac{d\hat{k}'}{dt} = \vec{\Omega} \times \hat{k}'$

$$\left(\frac{d\vec{A}}{dt}\right)_{inertial} \qquad \left(\frac{d\vec{A}}{dt}\right)_{rotating} \qquad \begin{matrix}(\textit{effects of rotation})\\ \vec{\Omega}\times\vec{A}\end{matrix}$$

$$\frac{dA_x}{dt}\hat{i}+\frac{dA_y}{dt}\hat{j}+\frac{dA_z}{dt}\hat{k}=\frac{dA'_x}{dt}\hat{i}'+\frac{dA'_y}{dt}\hat{j}'+\frac{dA'_z}{dt}\hat{k}'+A'_x\left(\vec{\Omega}\times\hat{i}'\right)+A'_y\left(\vec{\Omega}\times\hat{j}'\right)+A'_z\left(\vec{\Omega}\times\hat{k}'\right)$$

$$\left(\frac{d\vec{A}}{dt}\right)_{inertial}=\left(\frac{d\vec{A}}{dt}\right)_{rotating}+\vec{\Omega}\times\vec{A}$$

This equation provides us with a formal way of expressing the balance of forces on a fluid parcel in a rotating coordinate system.

Newton's second law in an inertial reference frame:

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{\sum \vec{F}}{m}$$

To transform to rotating coordinates:

$$\left(\frac{d\vec{r}}{dt}\right)_{inertial} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\Omega} \times \vec{r}$$

$\vec{r}$ is the position vector for an air parcel on the rotating earth

$$\vec{V}_{inertial} = \vec{V} + \vec{\Omega} \times \vec{r}$$

Velocity is the rate of change of the position vector with time

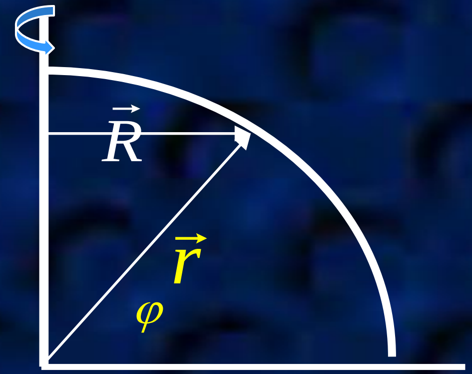
$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d\vec{V}_{inertial}}{dt} + \vec{\Omega} \times \vec{V}_{inertial}$$

Using the transformation of the total derivative

$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d\vec{V}_{inertial}}{dt} + \vec{\Omega}\times\vec{V}_{inertial}$$

$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d}{dt}\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)+\vec{\Omega}\times\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)$$

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{\Omega}\times(\vec{\Omega}\times\vec{R}) = -\Omega^2\vec{R}$$



$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R}$$

Using some vector identities and defining $\vec{R}$ as a vector perpendicular to the axis of rotation with magnitude equal to the distance to the axis of rotation.

Acceleration following the motion in an inertial system

Rate of change of relative velocity following the relative motion in a rotating reference frame.

Coriolis acceleration

Centrifugal acceleration

Substituting into Newton's second law:

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{\sum \vec{F}}{m}$$

$$\frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} = \frac{\sum \vec{F}}{m}$$

If the real forces acting on a fluid parcel are the pressure gradient force, gravitation and friction, then

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Rate of change of relative velocity following the relative motion in a rotating reference frame.

Coriolis acceleration

Pressure gradient force (per unit mass)

Gravity term (gravitation + centrifugal)

Friction

Vector momentum equation in rotating coordinates

Momentum Equations in Spherical Coordinates

For a variety of reasons, it is useful to express the vector momentum equation for a rotating earth as a set of scalar component equations.

The use of latitude-longitude coordinates to describe positions on earth's surface makes it convenient to write the momentum equations in spherical coordinates.

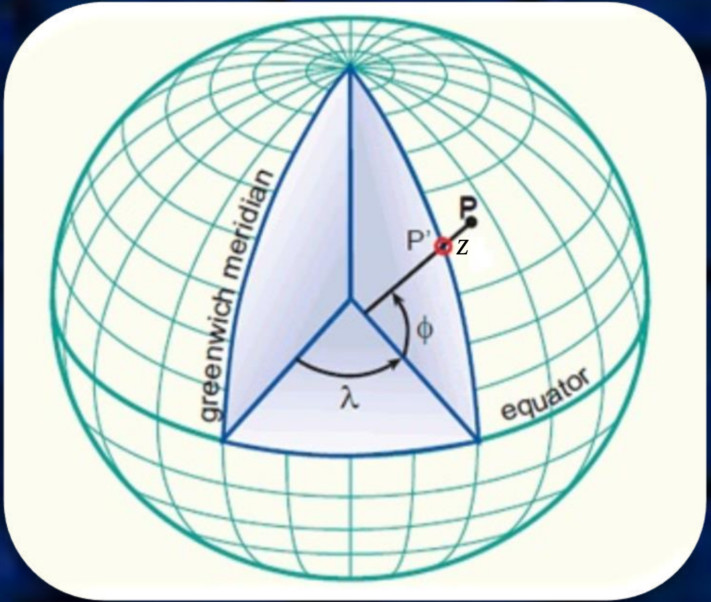
The coordinate axes are (λ, ϕ, z)

Where,

λ is longitude,

ϕ is latitude,

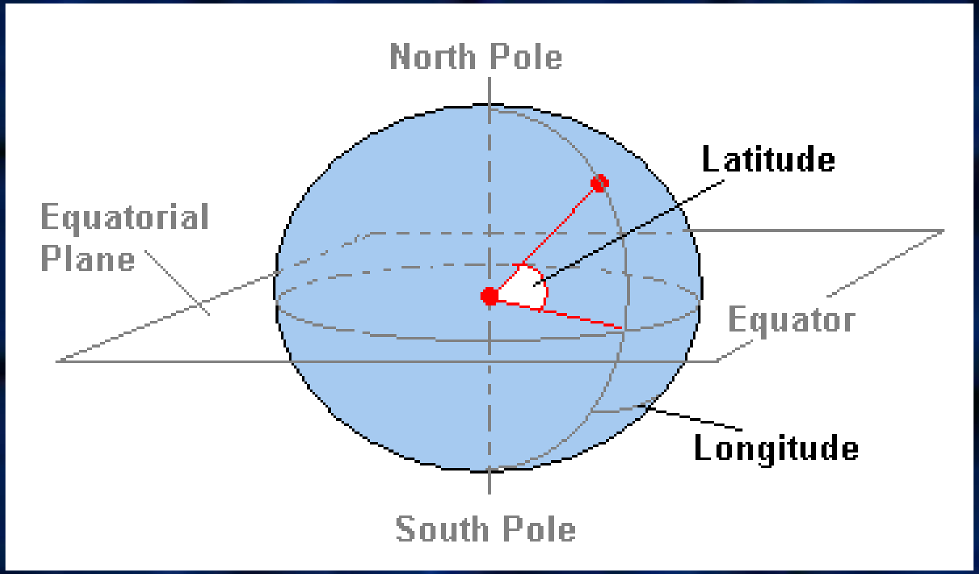
z is height.



Orientation of Coordinate Axes

The x- and y-axes are customarily defined to point east and north, respectively, such that

$$dx = Rd\lambda = a\cos\phi\, d\lambda$$



$$dy = a\, d\phi$$

$$dz = dr \qquad r=a+z$$

Thus the velocity components are

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dr}{dt}$$

$$\frac{dx}{dt} = a\cos\varphi\frac{d\lambda}{dt} \qquad \frac{dy}{dt} = a\frac{d\varphi}{dt} \qquad \frac{dz}{dt} = w$$

The unit vectors in the spherical coordinate system are functions of position

The (x,y,z) coordinates system defined in this way is not a Cartesian coordinates system , because the directions of the unit vectors depend on their position on the earth's surface.

This position dependence of the unit vectors must be taken into account when the acceleration vector is expended into its components on the sphere. Thus, we write:

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

We need to determine what these are