# Geophysical Fluid Dynamics

## Lecture 7

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## Geopotential Height

Definition:

The geopotential height (Z) is the actual height normalized by the globally averaged acceleration due to gravity at the Earth's surface ($g_0$ = 9.81 m $s^{-2}$), and is defined by:

$$Z = \frac{\Phi(z)}{g_0}$$

Used as the vertical coordinate in most atmospheric applications in which energy plays an important role (i.e. just about everything)

We can define a quantity called the geopotential, which is related to gravity.

Gravity can be represented as the gradient of the geopotential.

$$\nabla \Phi = -\vec{g}$$

Because $\vec{g} = -g\hat{k},$ then $\Phi = \Phi(z), \quad \dfrac{d\Phi}{dz} = g$

If the value of the geopotential is set to zero at mean sea level, the geopotential $\Phi(z)$ at height z is the work required to raise a unit mass to height z from mean sea level:

$$\Phi = \int_0^z g\, dz$$

Units of geopotential are J $kg^{-1}$, which are equivalent to $m^2\ s^{-2}$.

$g_o$ is a constant (9.8 $ms^{-2}$). In the lower atmosphere, Z is very close to z (called the 'geometric height'). The table below shows how Z, z, and g vary with height at a typical mid-latitude location.

$$Z = \frac{\Phi(z)}{g_0} = \frac{gz}{980cms^{-1}}$$

| z(km) | Z(km) | g($ms^{-2}$) |
|---|---|---|
| 0 | 0 | 9.802 |
| 1 | 1.000 | 9.798 |
| 10 | 9.986 | 9.771 |
| 20 | 19.941 | 9.741 |
| 30 | 29.864 | 9.710 |
| 60 | 59.449 | 9.620 |
| 90 | 88.758 | 9.531 |
| 120 | 117.795 | 9.443 |
| 160 | 156.096 | 9.327 |
| 200 | 193.928 | 9.214 |
| 300 | 286.520 | 8.940 |
| 400 | 376.370 | 8.677 |
| 500 | 463.597 | 8.427 |
| 600 | 548.314 | 8.186 |

$$Z = \frac{\Phi(z)}{g_0} = \frac{gz}{980cms^{-1}}$$

# Geopotential Height

- ✓ Earth's gravity has Altitude and Latitude dependences.
- ✓ Overall these are minor (few tenths of a percent under conditions of interest to meteorologist).
- ✓ Geopotential Height allows us a new reference (or coordinate) to accommodate these changes in gravity.
- ✓ Geopotential height is a vertical coordinate referenced to Earth's mean sea level — an adjustment to geometric height (elevation above mean sea level) using the variation of gravity with latitude and elevation.
- ✓ Thus it can be considered a "gravity-adjusted height".

Geometric vs. Geopotential Meters

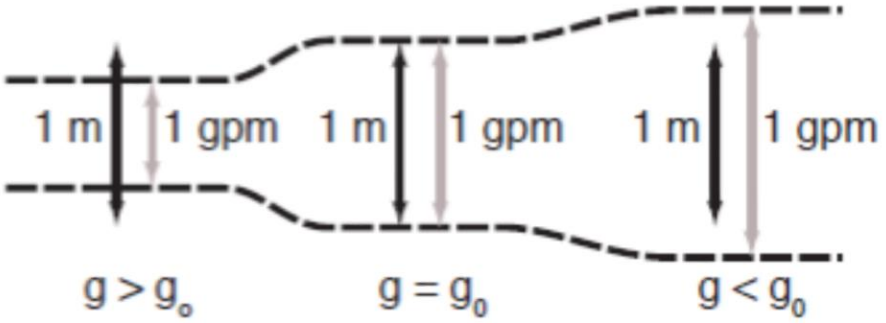
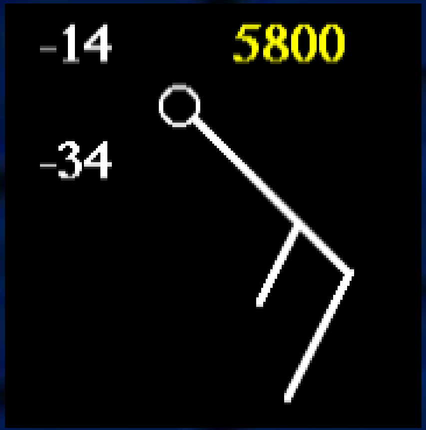



Fig. 4.2

Geopotential => Potential energy per unit mass

# Geopotential Height
## upper air station reports

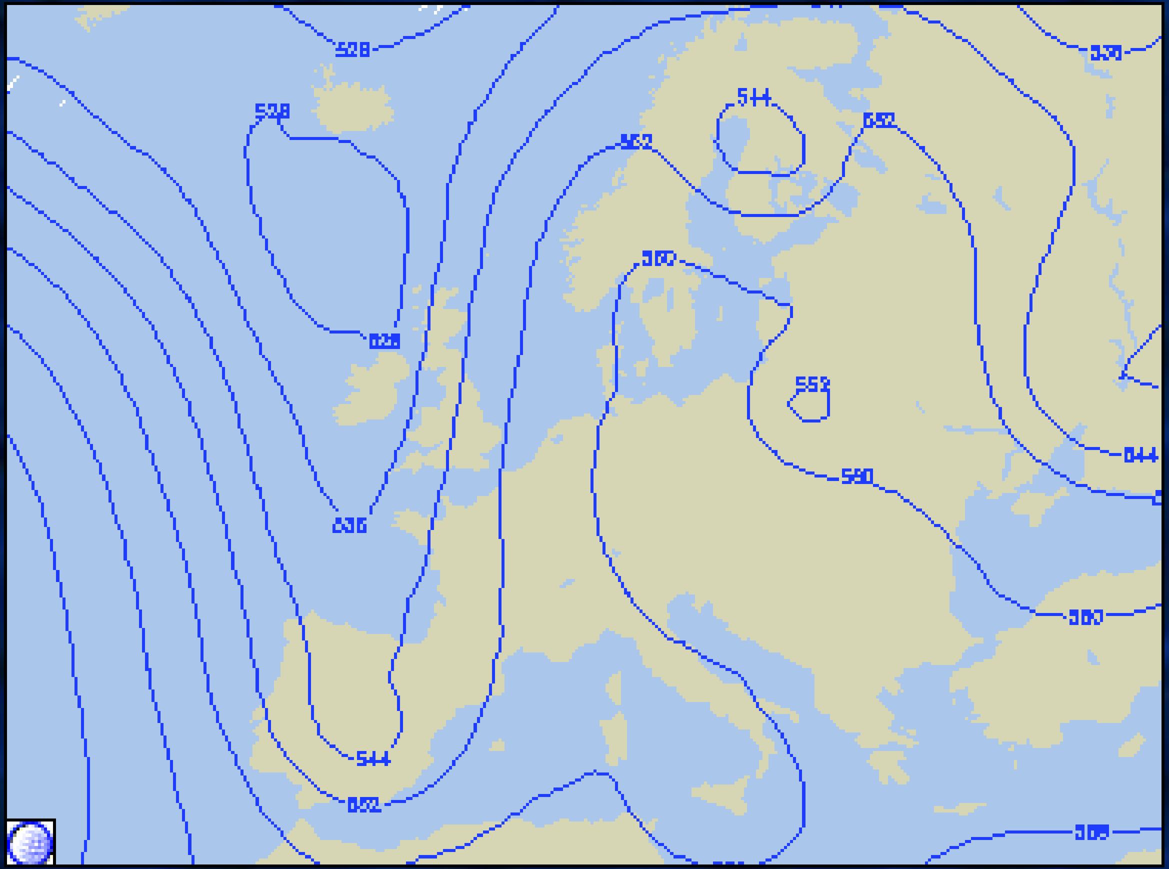



The value highlighted in yellow located in the upper right corner (in the diagram above) represents the geopotential height of a given pressure surface in meters (as reported by weather balloons).
Geopotential Height approximates the actual height of a pressure surface above mean sea-level. Therefore, for the example given above, the height of the pressure surface on which the observation was taken is 5800 meters.

When a collection of geopotential height reports are contoured on a given pressure surface, we are able to identify upper air troughs and ridges, which are very important influences on surface weather conditions.

# Geopotential height 500 hPa [gpdm]

## Fri 10 Apr, 16:00 BST (15:00 UTC)

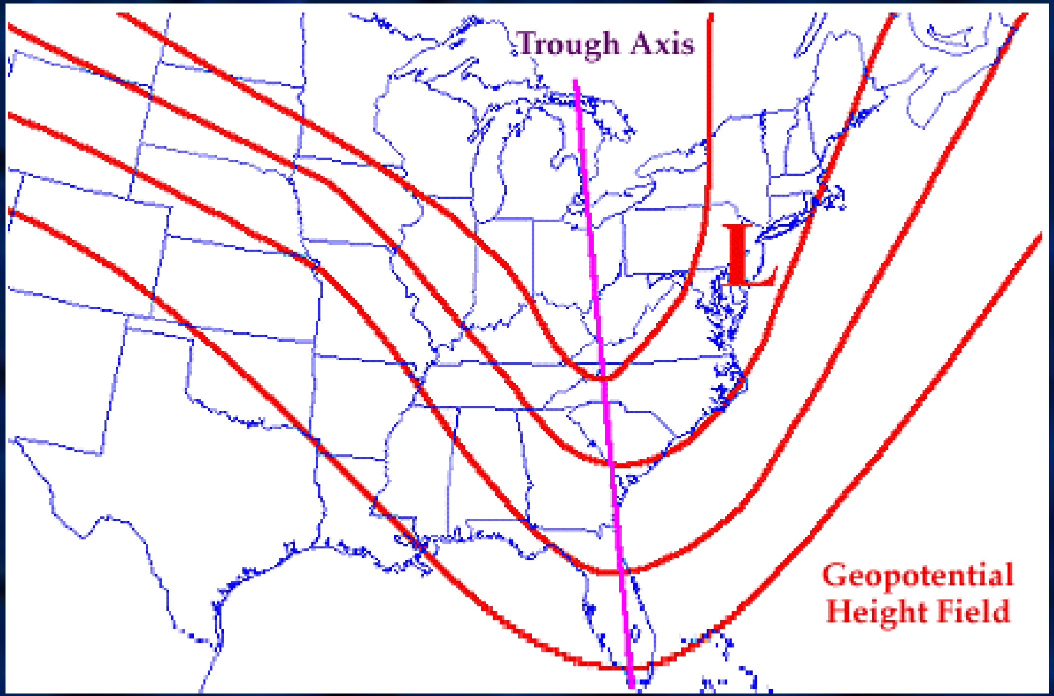



http://www.weatheronline.co.uk/

# Troughs
## upper level lows

When the height contours bend strongly to the south, (as in the diagram below), it is called a TROUGH.

Strong troughs are typically preceded by stormy weather and colder air at the surface. Below is an example of a trough in an upper-level height field (red contours).
The trough axis is denoted by the purple line.



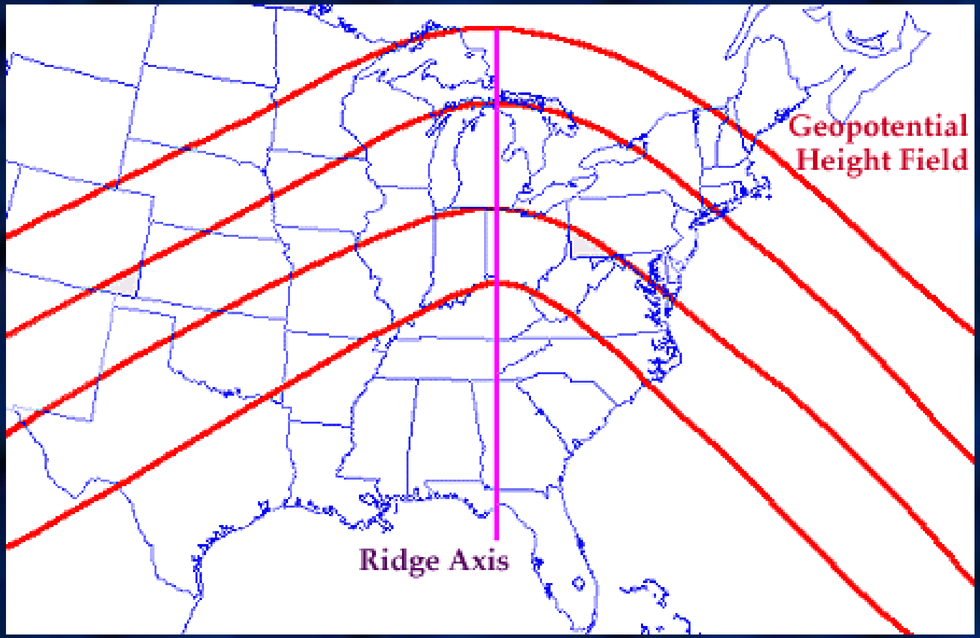
# Ridges

## upper level highs

When the height contours bend strongly to the north (as in the diagram below), this is known as a RIDGE.

Strong ridges are accompanied by warm and dry weather conditions at the surface. Below is an example of a ridge in an upper-level height field (red contours).

The purple line denotes the ridge axis.



## Hypsometric Equation

Derivation:

If we combine the Hydrostatic Equation with the Ideal Gas Law for moist air and the Geopotential Height, we can derive an equation that defines the thickness of a layer between two pressure levels in the atmosphere

1. Substitute the ideal gas law into the Hydrostatic Equation

$$\frac{dp}{dz} = -\rho g$$

$$p = \rho R_d T_v$$

$$\frac{dp}{dz} = \frac{-p\, g}{R_d T_v}$$

**Virtual Temperature:** The temperature that a parcel of dry air would have if it were at the same pressure and had the same density as moist air.

P = pressure
$\rho_d$ = dry air density
$\rho_v$ = vapor density
ρ= air density
R= gas constant
$R_v$ = vapor gas constant
$R_d$ = dry air gas constant
T = Temperature

**Derivation:**

Start with ideal gas law for moist air:

$$P = \rho R T$$

$$P = \rho_d R_d T + \rho_v R_v T$$

Now treat moist air as if it were dry by introducing the virtual temperature $T_v$

$$P = (\rho_d R_d + \rho_v R_v) T = (\rho_d + \rho_v) R_d T_v = \rho R_d T_v$$

What is the relationship between the temperature, T and the virtual temperature $T_v$?

$$T_v = (1 + 0.61 r_v) T$$

2. Re-arranging the equation and using the definition of geopotenital height:

$$\frac{dp}{dz} = \frac{-p\,g}{R_d T_v}$$

$$d\Phi = g dz = -R_d T_v \frac{dp}{p}$$

3. Integrate this equation between two geopotential heights ($\Phi_1$ and $\Phi_2$) and the two corresponding pressures ($p_1$ and $p_2$), assuming $T_v$ is constant in the layer

$$\int_{\Phi_1}^{\Phi_2} d\Phi = -R_d T_v \int_{p_1}^{p_2} \frac{dp}{p}$$

4. Performing the integration:

$$\int_{\Phi_1}^{\Phi_2} d\Phi = -R_d T_v \int_{p_1}^{p_2} \frac{dp}{p}$$

$$\Phi_2 - \Phi_1 = -R_d T_v \ln\left(\frac{p_2}{p_1}\right)$$

5. Dividing both sides by the gravitational acceleration at the surface ($g_0$):

$$\frac{\Phi_2}{g_0} - \frac{\Phi_1}{g_0} = -\frac{R_d T_v}{g_0} \ln\left(\frac{p_2}{p_1}\right)$$

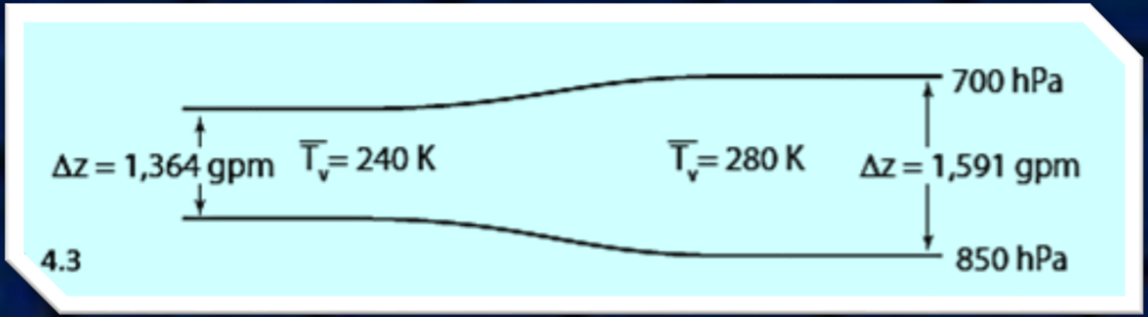
6. Using the definition of geopotential height

$$\frac{\Phi_2}{g_0} - \frac{\Phi_1}{g_0} = -\frac{R_d T_v}{g_0} \ln\left(\frac{p_2}{p_1}\right)$$

$$Z_2 - Z_1 = -\frac{R_d T_v}{g_0} \ln\left(\frac{p_2}{p_1}\right)$$ Hypsometric Equation

Defines the geopotential thickness ($Z_2 - Z_1$) between any two pressure levels ($p_1$ and $p_2$) in the atmosphere.

The layer thickness ($\Delta z$) is directly proportional to mean virtual temperature ($T$v)



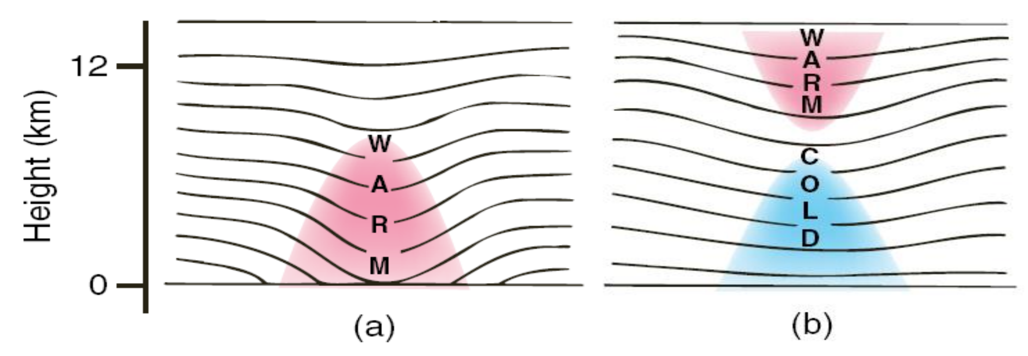
Interpretation:

The thickness of a layer between two pressure levels is proportional to the mean virtual temperature of that layer.

$$Z_2 - Z_1 = -\frac{R_d T_v}{g_0} \ln\left(\frac{p_2}{p_1}\right)$$

If $T_v$ increases, the air between the two pressure levels expands and the layer becomes thicker. Pressure decrease slowly with height

If $T_v$ decreases, the air between the two pressure levels compresses and the layer becomes thinner. Pressure decreases rapidly with height




Black solid lines are pressure surfaces

Hurricane (warm core)    Mid-latitude Low (cold core)

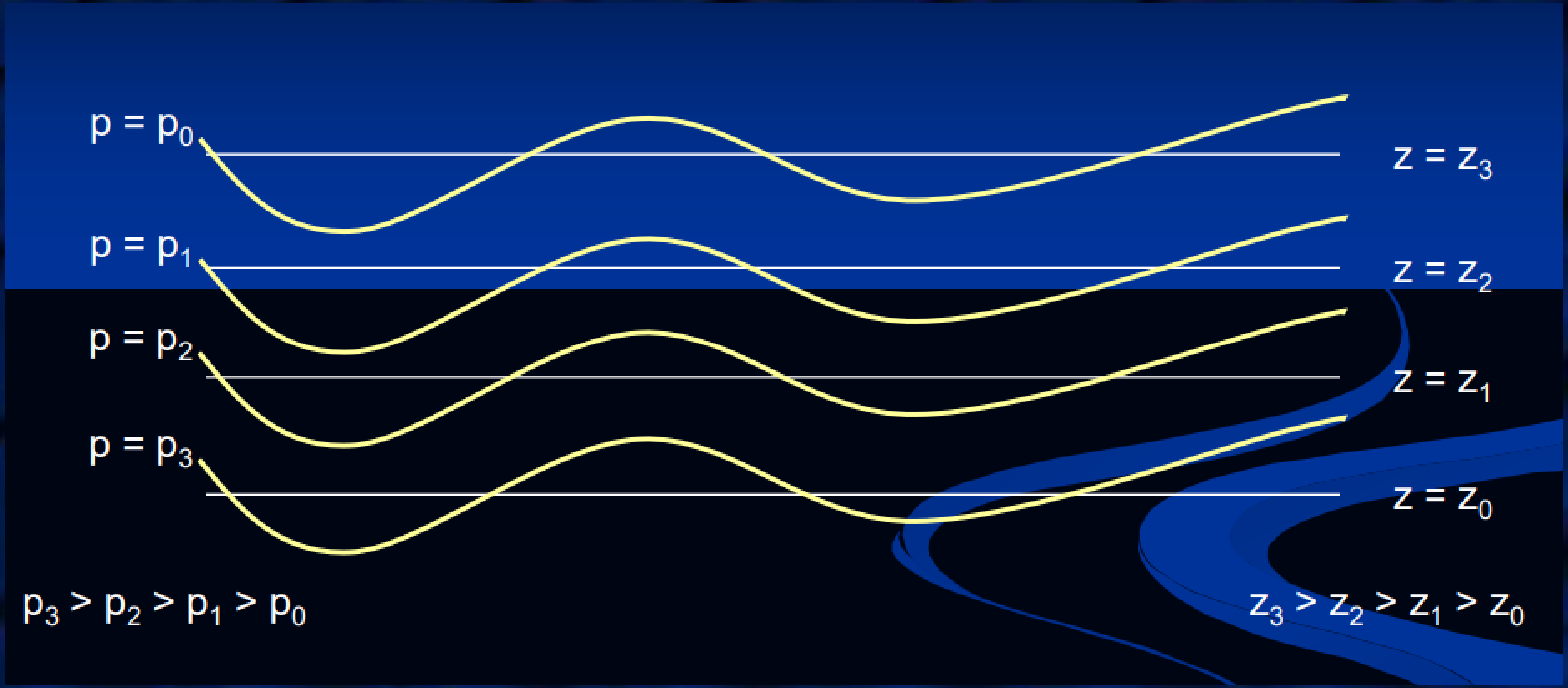
## Generalized Vertical Coordinates

The use of pressure as a vertical coordinate is a specific example of the use of generalized vertical coordinates.

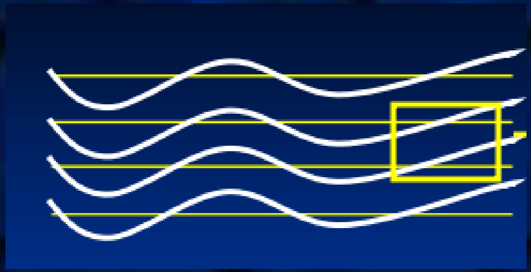
Any quantity $s = s(x,y,z,t)$ that changes monotonically with height can be used as a vertical coordinate.

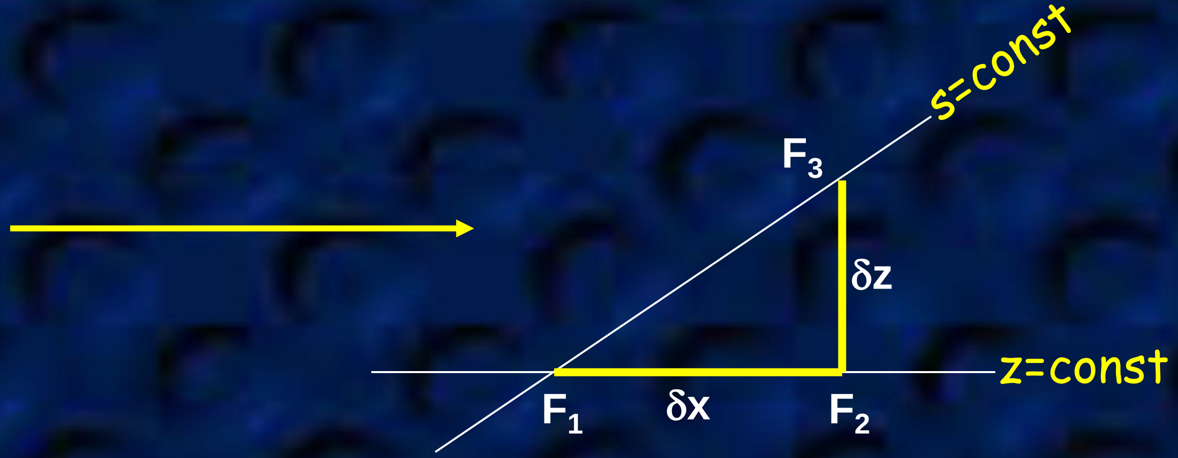
If we wish to transform equations from $(x,y,z)$ coordinates to $(x,y,s)$ coordinates, derivatives must be transformed.

# Pressure As A Vertical Coordinate




How do we convert our equations from height coordinates (x,y,z) to pressure coordinates (x,y,p)?






Let F = some scalar property,
and
s = a generalized vertical coordinate.

We would like to transform derivatives such as

$$\left(\frac{\partial F}{\partial x}\right)_z \quad \text{to} \quad \left(\frac{\partial F}{\partial x}\right)_s$$

Derivative in x-direction on a constant z surface

Derivative in x-direction on a constant s surface

$$\frac{F_3 - F_1}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta z}\frac{\delta z}{\delta x}$$

$$\left(\frac{\delta F}{\delta x}\right)_s = \left(\frac{\delta F}{\delta x}\right)_z + \frac{\delta F}{\delta z}\left(\frac{\delta z}{\delta x}\right)_s$$

$$\begin{cases} \left(\dfrac{\partial F}{\partial x}\right)_s = \left(\dfrac{\partial F}{\partial x}\right)_z + \dfrac{\partial F}{\partial z}\left(\dfrac{\partial z}{\partial x}\right)_s \\ \left(\dfrac{\partial F}{\partial y}\right)_s = \left(\dfrac{\partial F}{\partial y}\right)_z + \dfrac{\partial F}{\partial z}\left(\dfrac{\partial z}{\partial y}\right)_s \end{cases}$$

$$\nabla_s F = \left(\frac{\partial F}{\partial x}\right)_s \hat{i} + \left(\frac{\partial F}{\partial y}\right)_s \hat{j}$$

$$\nabla_z F = \left(\frac{\partial F}{\partial x}\right)_z \hat{i} + \left(\frac{\partial F}{\partial y}\right)_z \hat{j}$$

$$\nabla_s z = \left(\frac{\partial z}{\partial x}\right)_s \hat{i} + \left(\frac{\partial z}{\partial y}\right)_s \hat{j}$$

*can be written in vector form as*

$$\boxed{\nabla_s F = \nabla_z F + \frac{\partial F}{\partial z}\nabla_s z}$$

We will use this equation to transform the horizontal derivatives in the momentum equation from z-coordinates to p-coordinates.