Geophysical Fluid Dynamics

Lecture 20

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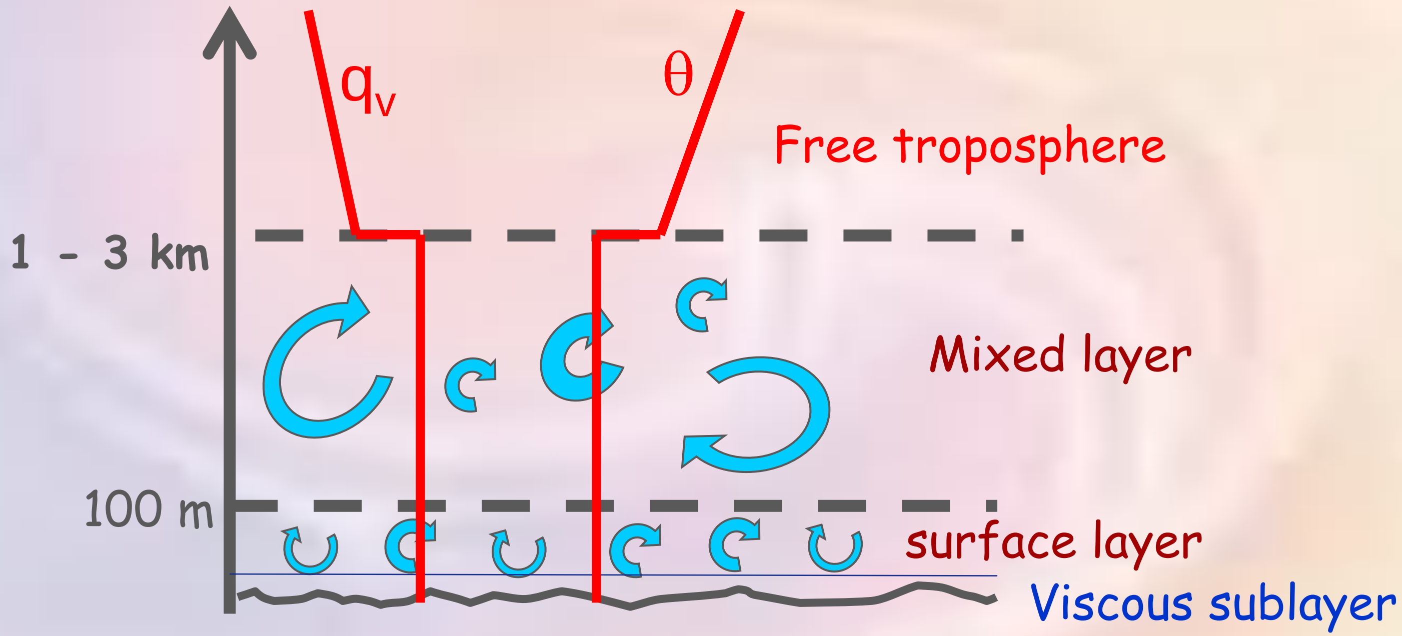
Atmospheric Planetary Boundary Layer

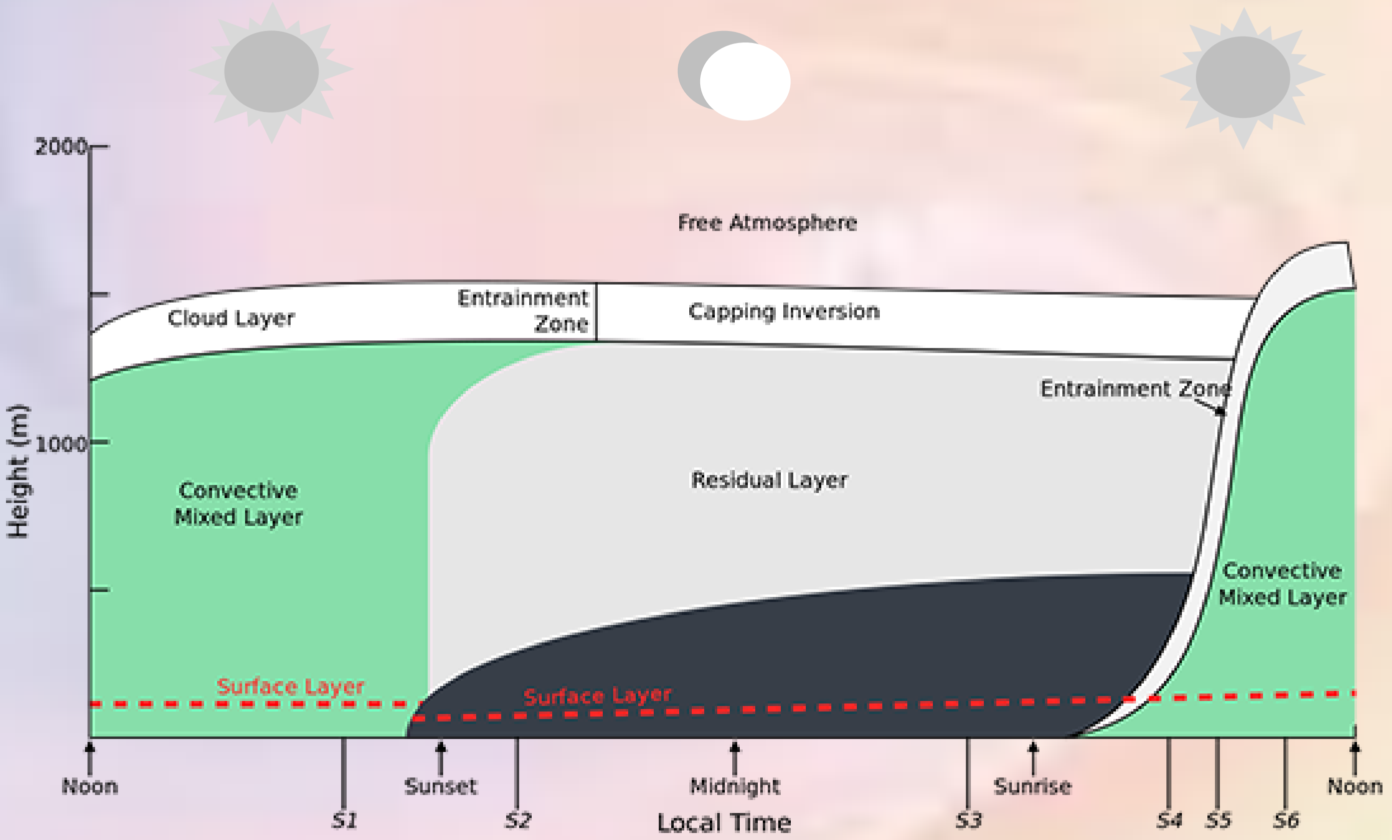
The layer of air influenced by surface friction is called the planetary boundary layer (PBL).



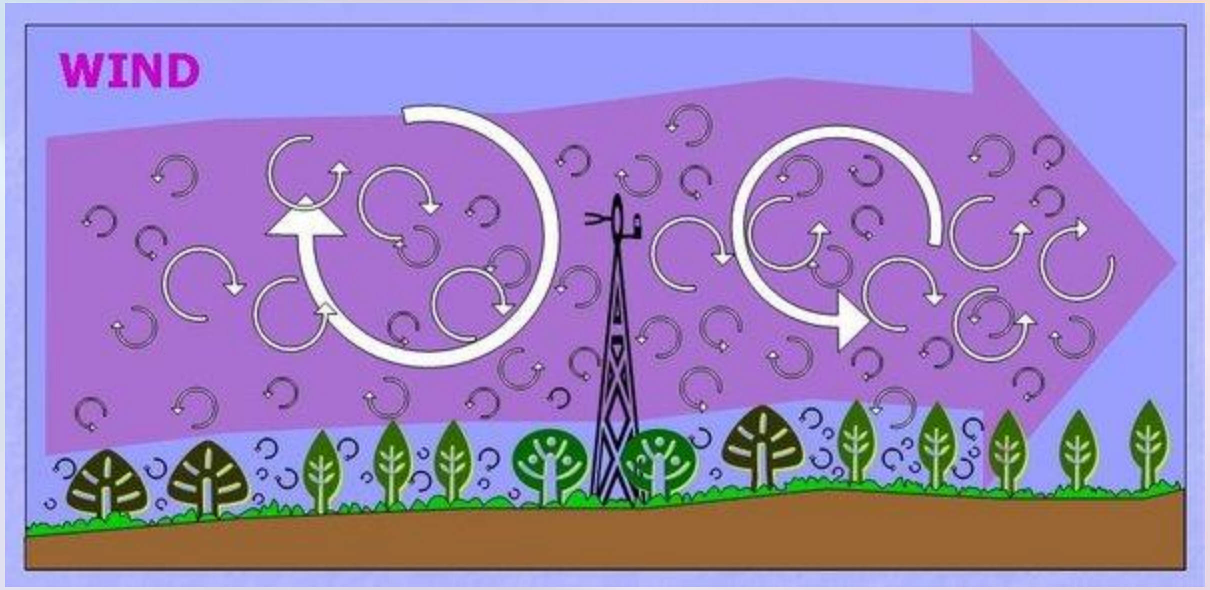
The atmosphere above the boundary layer is called the free troposphere, typically.

Planetary Boundary Layer

Diurnal Variability



Planetary Boundary Layer



In this section we will examine the effect of friction on the wind in the boundary layer (Ekman Spiral), and learn how friction always leads to the demise of high and low pressure systems.

Laminar and Turbulent Flows

The Reynolds number (Re) is generally used to quantify whether a flow is laminar or turbulent:

$$\mathrm{Re} = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{UL}{\nu} = \frac{10 m/s \times 10^3 m}{1.45\times 10^{-5}} = 10^9$$

A critical value of Re between laminar and turbulent flows is ~3000.

Re = 10^9 >> 3000

Atmospheric flows are turbulent

General circulation of the oceans

Ocean surface currents – horizontal water motions

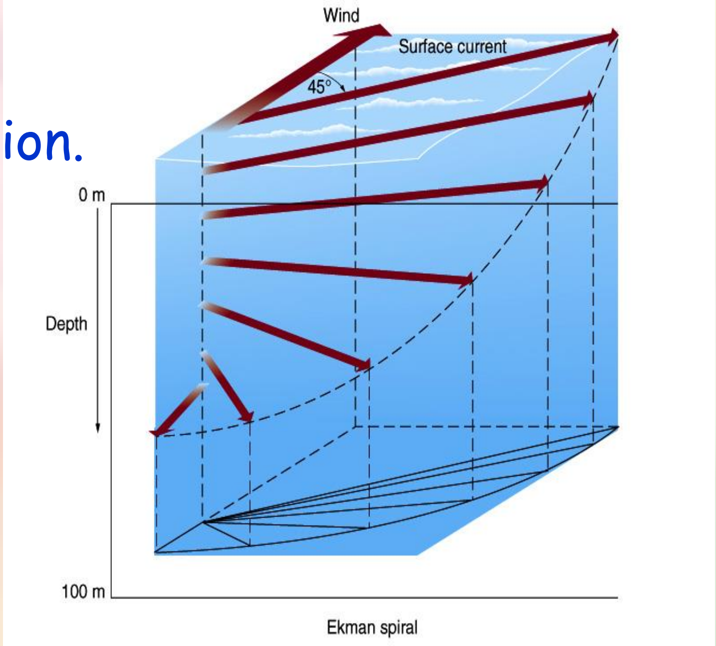
Transfer energy and influence overlying atmosphere

Surface currents result from frictional drag caused by wind - Ekman Spiral

As the surface layer of the ocean flows, it interacts with the water below it and transfer energy down, again by friction

Water moves at a 45^{o} angle (right) in N.H. to prevailing wind direction.

Greater angle at depth



Winds in the Free Troposphere

The acceleration of an air parcel is proportional to the pressure gradient.

Therefore, the wind results from a pressure difference between two locations: a steep pressure gradient leads to a strong wind due to the acceleration that results from that pressure gradient.

The direction of the air flow is, in its simplest form, from the high pressure area toward the low pressure area.

However, there are other forces (true or virtual) that affect wind direction.

Geostrophic Wind

$$-fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad fu = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

Horizontal Momentum Equations

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + fv \qquad \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - fu$$

Let us assume that:

(1) the flow is steady state

(2) for simplicity that the flow above the boundary layer is west-east

(3) the flow above the PBL is geostrophic

(4) K is constant (actually varies with z)

(5) f is constant

Making use of the geostrophic relationships:

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \qquad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

With these assumptions, our equations become

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + fv \longrightarrow 0 = K\frac{\partial^2 u}{\partial z^2} + f\left(v - v_g\right) \qquad (1)$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - fu \longrightarrow 0 = K\frac{\partial^2 v}{\partial z^2} - f\left(u - u_g\right) \qquad (2)$$

We will now do a mathematical trick to solve for the Ekman spiral

Lets multiply (2) by the imaginary number $i = \sqrt{-1}$ and add it to (1)

$$K\frac{\partial^2\left(u + iv\right)}{\partial z^2} - if\left(u + iv\right) = -if\left(u_g + iv_g\right)$$

This is a second order inhomogeneous ordinary differential equation

We will solve the equation with boundary conditions:

$$u = 0, \quad v = 0, \quad at \quad z = 0 \qquad\qquad u \to u_g, \; v \to v_g, \quad at \quad z = \infty$$

$$K\frac{\partial^2 (u+iv)}{\partial z^2} - if\,(u+iv) = -if\,(u_g + iv_g)$$

Let's simplify the look of the equation by using

$$N = u + iv \qquad N_g = u_g + iv_g \qquad A = \frac{if}{K}$$

$$\frac{\partial^2 N}{\partial z^2} - \frac{if}{K} N = -\frac{if}{K} N_g$$

This is a simple second order inhomogeneous ordinary differential equation

To find the general solution, we must find a single particular solution to the equation and a complimentary solution to the corresponding homogeneous equation:

$$\frac{\partial^2 N}{\partial z^2} - \frac{if}{K} N = 0$$

Let's seek a particular solution first:

$$\frac{\partial^2 N}{\partial z^2} - \frac{if}{K} N = -\frac{if}{K} N_g$$

Clearly, one solution of the inhomogeneous equation is obtained by assuming that N is independent of z. This reduces the equation to

$$N = N_g$$

Now let's seek a complementary solution to the homogeneous equation

$$\frac{\partial^2 N}{\partial z^2} - \frac{if}{K} N = 0$$

If we seek a solution of the form $N = A\, e^{\lambda z}$ we get

$$\lambda^2 = \frac{if}{K}$$

Then λ can have two values:

$$\lambda^2 = \frac{if}{K} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$\lambda_+ = \frac{1+i}{\sqrt{2}}\sqrt{\frac{f}{K}} \qquad \lambda_- = \frac{-1-i}{\sqrt{2}}\sqrt{\frac{f}{K}}$$

Let's define

$$\gamma = \sqrt{\frac{f}{2K}}$$

$$\lambda_+ = (1+i)\gamma \qquad \lambda_- = -(1+i)\gamma$$

The general solution to the homogeneous equation is therefore

$$N = A\,\mathrm{e}^{\lambda_+ z} + B\,\mathrm{e}^{\lambda_- z}$$

$$u + iv = A\,\mathrm{e}^{\lambda_+ z} + B\,\mathrm{e}^{\lambda_- z}$$

The first term on the right, $A\,e^{\lambda_+ z} \to \infty \;\; as \;\; z \to \infty$

The only way the solution can be finite is for $A = 0$

$$u + iv = B\,e^{-(1+i)\gamma z}$$

$$u + iv = B\,e^{-\gamma z}\,e^{-i\gamma z}$$

The complete solution is the sum of the particular solution to the inhomogeneous equation and the general solution to the homogeneous equation

$$u + iv = B\,e^{-\gamma z}\,e^{-i\gamma z} + (u_g + iv_g)$$

$$u + iv = B\,e^{-\gamma z}\,e^{-i\gamma z} + (u_g + iv_g)$$

We will solve the equation with boundary conditions:

$$u = 0, \quad v = 0, \quad at \;\; z = 0$$

$$u \rightarrow u_g, \quad v \rightarrow v_g, \quad at \;\; z = \infty$$

Setting z = 0

$$0 = B + (u_g + iv_g)$$

$$B = -(u_g + iv_g)$$

$$u + iv = -(u_g + iv_g)\,e^{-\gamma z}\,e^{-i\gamma z} + (u_g + iv_g)$$

$$u + iv = (u_g + iv_g)(1 - e^{-\gamma z}\,e^{-i\gamma z})$$