Geophysical Fluid Dynamics

Lecture 15

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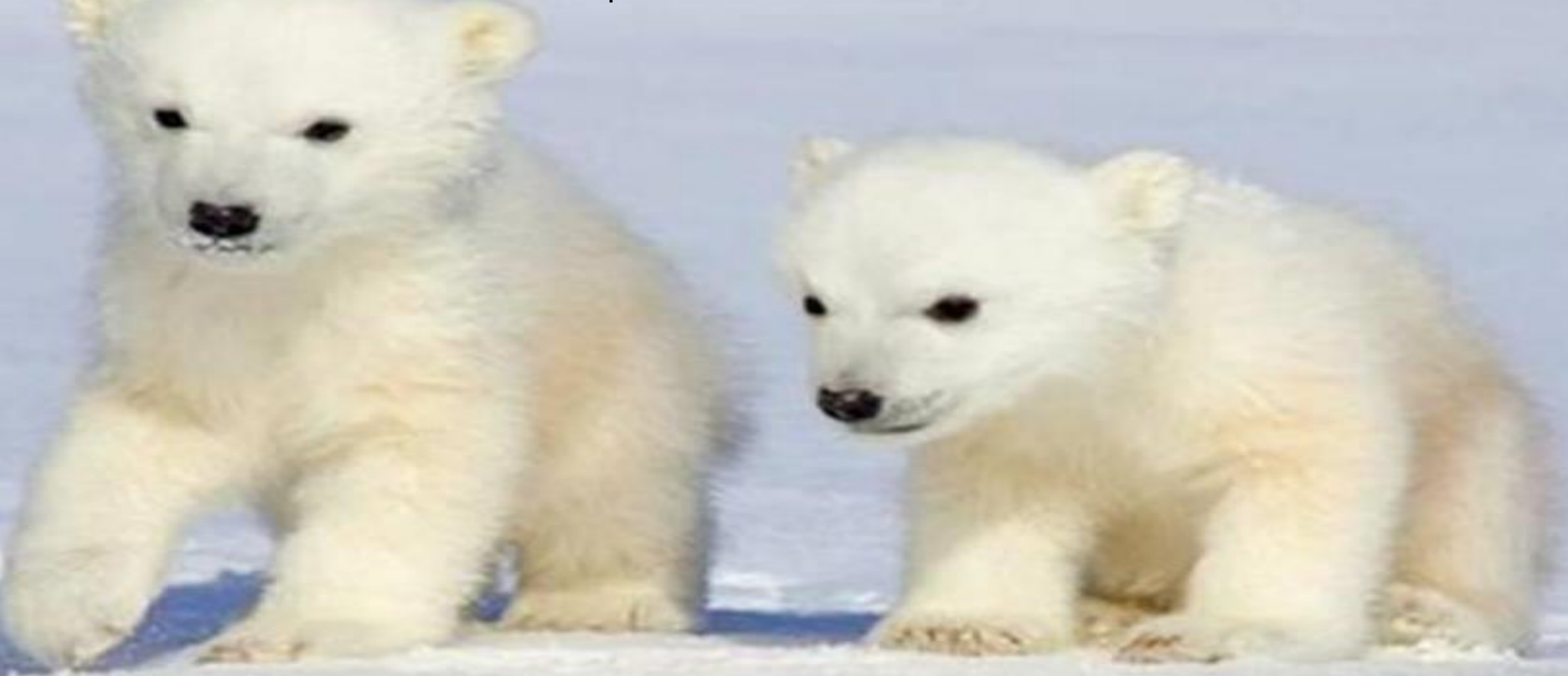

$$c_p \frac{d\ln T}{dt} - R\frac{d\ln p}{dt} = \frac{\dot{q}}{T} \equiv \frac{ds}{dt}$$

Potential Temperature

For an ideal gas that is undergoing an *adiabatic* process (i.e., a reversible process in which no heat is exchanged with the surroundings), the first law of thermodynamics can be written in differential form as

$$c_p d\ln T - R d\ln p = d(c_p \ln T - R\ln p) = 0$$

$$\theta = T(p_s / p)^{R/c_p}$$

Taking the logarithm of equation and differentiating, we find that

$$c_p \frac{d\ln\theta}{dt} = c_p \frac{d\ln T}{dt} - R\frac{d\ln p}{dt}$$

Comparing

$$\begin{cases} c_p \dfrac{d \ln T}{dt} - R \dfrac{d \ln p}{dt} = \dfrac{\dot{q}}{T} \equiv \dfrac{ds}{dt} \\ c_p \dfrac{d \ln \theta}{dt} = c_p \dfrac{d \ln T}{dt} - R \dfrac{d \ln p}{dt} \end{cases}$$

$$\Rightarrow c_p \frac{d \ln \theta}{dt} = \frac{\dot{q}}{T} = \frac{ds}{dt}$$

Thus, for reversible processes, changes in fractional potential temperature are indeed proportional to entropy changes.

A parcel that conserves entropy following the motion must move along an isentropic (constant θ) surface.

Scale Analysis of the Thermodynamic Energy Equation

$$c_p \frac{d \ln \theta}{dt} = \frac{\dot{q}}{T} = \frac{ds}{dt}$$

$$\theta_{tot} = \theta_0(z) + \theta'(x, y, z, t)$$

the first law of thermodynamics can be written approximately for synoptic scaling as

$$c_p \frac{d \ln \theta}{dt} = \frac{\dot{q}}{T} = \frac{ds}{dt}$$

$$\frac{1}{\theta}\frac{d\theta}{dt} = \frac{\dot{q}}{c_p T} \qquad \frac{1}{\theta_0 + \theta'}\frac{d}{dt}(\theta_0 + \theta') = \frac{\dot{q}}{c_p T} \qquad \frac{1}{\theta_0(1 + \frac{\theta'}{\theta_0})}(\frac{d\theta_0}{dt} + \frac{d\theta'}{dt}) = \frac{\dot{q}}{c_p T}$$

$$\frac{1}{\theta_0}\left(w\frac{\partial \theta_0}{\partial z}+\frac{\partial \theta'}{\partial t}+\vec{V}.\nabla\theta'\right)=\frac{\dot{q}}{c_p T}$$

$$\frac{1}{\theta_0}\left(\frac{\partial \theta'}{\partial t}+\vec{V}.\nabla\theta'\right)+w\frac{d\ln\theta_0}{\partial z}=\frac{\dot{q}}{c_p T}$$

$$\frac{1}{\theta_0}\left(\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+v\frac{\partial \theta'}{\partial y}\right)+w\frac{d\ln\theta_0}{\partial z}=\frac{\dot{q}}{c_p T}$$

where we have used the facts that for

$$\left|\theta'/\theta_0\right| << 1 \qquad \left|d\theta'/dz\right| << d\theta_0/dz \qquad \ln\theta_{tot}=\ln\left[\theta_0(1+\theta'/\theta_0)\right]\approx\ln\theta_0+\theta'/\theta_0$$

$$\frac{1}{\theta_0}\left(\frac{\partial\theta'}{\partial t}+u\frac{\partial\theta'}{\partial x}+v\frac{\partial\theta'}{\partial y}\right)+w\frac{d\ln\theta_0}{\partial z}=\frac{\dot{q}}{c_pT}$$

In the troposphere, radiative heating is quite weak so that typically $\frac{\dot{q}}{c_p}\leq 1\,^oC\,d^{-1}$

The typical amplitude of horizontal potential temperature fluctuations in a midlatitude synoptic system (above the boundary layer) is $\theta\approx 4\,^oC$. Thus,

$$\frac{T}{\theta_0}\left(\frac{\partial\theta'}{\partial t}+u\frac{\partial\theta'}{\partial x}+v\frac{\partial\theta'}{\partial y}\right)\approx\frac{\theta' U}{L}\approx 4\,^oC\,d^{-1}$$

$$w\left(\frac{T}{\theta_0}\frac{d\theta_0}{\partial z}\right)=w(\Gamma_d-\Gamma)\approx 4\,^oC\,d^{-1}$$

Thus, in the absence of strong diabatic heating, the rate of change in the perturbation potential temperature is equal to the adiabatic heating or cooling due to vertical motion in the statically stable basic state, and equation can be approximated as

$$\left(\frac{\partial\theta'}{\partial t}+u\frac{\partial\theta'}{\partial x}+v\frac{\partial\theta'}{\partial y}\right)+w\frac{d\theta_0}{\partial z}=0$$

$$T_{tot}=T_0(z)+T'(x,y,z,t)$$

$$\theta'/\theta_0\approx T'/T_0$$

The above equation can be expressed to the same order of approximation in terms of temperature as

$$\left(\frac{\partial T'}{\partial t}+u\frac{\partial T'}{\partial x}+v\frac{\partial T'}{\partial y}\right)+w(\Gamma_d-\Gamma)=0$$

THE BOUSSINESQ APPROXIMATION

$$\frac{du}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + fv + F_{rx} \qquad \frac{dv}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} - fu + F_{ry}$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(P_0 + P') - g = \frac{1}{\rho_0}(\rho' g + \frac{\partial P'}{\partial z}) = 0$$

the vertical momentum equation becomes

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g\frac{\theta'}{\theta_0} + F_{rz}$$

the adiabatic thermodynamic energy equation has a form similar to

$$\left(\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + v\frac{\partial \theta'}{\partial y}\right) + w\frac{d\theta_0}{\partial z} = 0$$

$$\frac{d\theta'}{dt} = -w\frac{d\theta_0}{\partial z}$$

except, from the full material derivative, we see that the vertical advection of perturbation potential temperature is formally included. Finally, the continuity equation under the Boussinesq approximation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Elementary Application of the Basic Equation

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In addition to the geostrophic wind, which was discussed in previous chapter there are other approximate expression for the relationships among the velocity, pressure, and temperature fields, which are useful in the analysis of weather systems.

These are most conveniently discussed using a coordinate system in which pressure is the vertical coordinate.

The basic Equations in Isobaric Coordinates

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

The approximate horizontal momentum equation may be written in vectorial form as

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = -\frac{1}{\rho}\nabla p \qquad \text{where} \quad \vec{V} = \hat{i}u + \hat{j}v$$

In order to express this equation in isobaric coordinate form the pressure gradiant force using previous equations to obtain:

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = -\nabla_p \Phi$$

$$dp/dz = -\rho g$$

$$\nabla_z p = \rho g \nabla_p z$$

$$-\frac{1}{\rho}\nabla_z p = -g\nabla_p z = -\nabla_p \Phi$$

Since p is the independent vertical coordinate we must expand the total derivative as follows:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

From momentum equation we see that the isobaric coordinate form of the geostrophic relationship is:

$$\vec{\mathbf{V}}_g \equiv \hat{\mathbf{k}} \times \frac{1}{\rho f} \nabla p$$

$$f\vec{V}_g = \hat{k} \times \nabla_p \Phi$$

The Continuity Equation

Isobaric form by considering a Lagrangian control volume

$$\delta V = \delta x \, \delta y \, \delta z$$

Applying the hydrostatic equation $\delta p = -\rho g \delta z$

$$\delta V = -\delta x \, \delta y \, \delta p / (\rho g)$$

$$\delta M = \rho \delta V = -\delta x \, \delta y \, \delta p / g$$

$$\frac{1}{\delta M}\frac{d}{dt}(\delta M) = \frac{g}{\delta x\,\delta y\,\delta p}\frac{d}{dt}\left(\frac{\delta x\,\delta y\,\delta p}{g}\right) = 0$$

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

The continuity equation in the isobaric system

The Thermodynamic Energy Equation

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{q}$$

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha \omega = \dot{q}$$