Geophysical Fluid Dynamics

Lecture 14

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The Thermodynamic Energy Equation

We now turn to the third fundamental conservation principle, the conservation of energy, as applied to a moving fluid element.

The first law of thermodynamics is usually derived by considering a system in thermodynamic equilibrium.

A system that is initially at rest and after exchanging heat with its surroundings and doing work on the surroundings is again at rest.

For such a system the first law states that the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system.

e: internal energy per unit mass

$\frac{1}{2}\vec{V}.\vec{V}$: Kinetic energy owing to the macroscopic motion of the fluid.

Total thermodynamic energy contained in a Lagrangian fluid element of density ρ and volume δV is:

$$\rho\left[e + (1/2)\vec{V}.\vec{V}\right]\delta V$$

The external forces that act on a fluid element may be divided into surface forces, such as pressure and viscosity, and body forces such as gravity or Coriolis force.

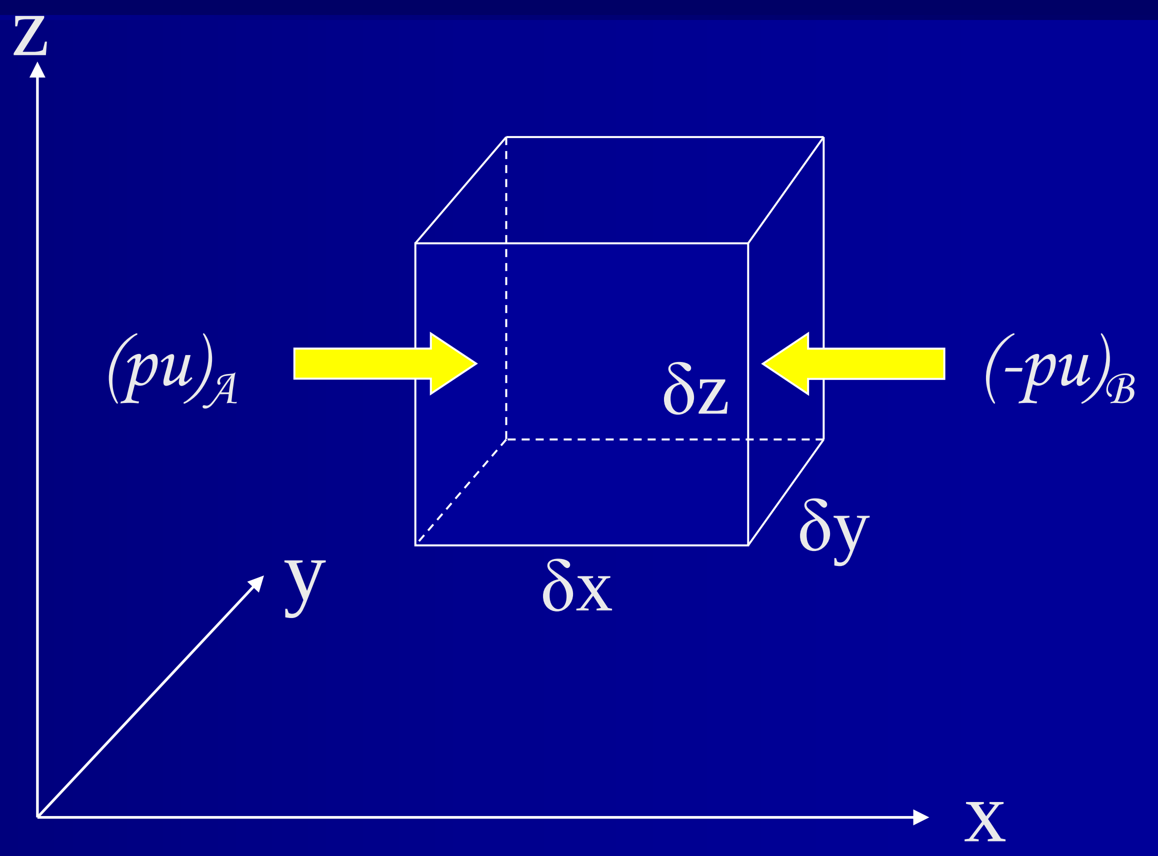
The rate at which work is done on the fluid element by the x component of the pressure force is illustrated in following Fig.

The rate at which the surrounding fluid does work on the element owing to the pressure force on the two boundary surfaces in the y,z plane is given by:

$$(pu)_A \delta y \delta z - (pu)_B \delta y \delta z$$

Now by expanding Taylor series we can write:

$$(pu)_B = (pu)_A + \left[\frac{\partial}{\partial x}(pu)\right]_A \delta x + \ldots$$



Thus the net rate at which the pressure force does work owing to the x component of motion is:

$$[(pu)_A - (pu)_B]\delta y \delta z = -\left[\frac{\partial}{\partial x}(pu)\right]_A \delta V$$

Similarly, we can show that the net rates at which the pressure force does work owing to the y and z components of motion are:

$$-\left[\frac{\partial}{\partial y}(pv)\right]_A \delta V \quad \textit{and} \quad -\left[\frac{\partial}{\partial z}(pw)\right]_A \delta V$$

Hence, the total rate at which work is done by the pressure force is simply

$$-\nabla.(p\overleftarrow{V})\delta V$$

The only body forces of meteorological significant that act on an element of mass in the atmosphere are the Coriolis force and gravity.

However, since the Coriolis force $-2\vec{\Omega}\times\vec{V}$ is perpendicular to the velocity vector it can do no work.

Thus the rate at which body forces do work on the mass element is just:

$$\rho\vec{g}.\vec{V}\delta V$$

Applying the principle of energy conservation to our Lagrangian control volume (neglecting effects of molecular viscosity), we thus obtain:

$$\frac{d}{dt}\left[\rho\left(e+\frac{1}{2}\vec{V}.\vec{V}\right)\delta V\right]=-\nabla.(p\vec{V})\delta V+\rho\vec{g}.\vec{V}\delta V+\rho\dot{q}\delta V$$

$$\rho\delta V\,\frac{d}{dt}\left(e+\frac{1}{2}\vec{V}.\vec{V}\right)+\left(e+\frac{1}{2}\vec{V}.\vec{V}\right)\frac{d(\rho\delta V)}{dt}$$

$$= -\vec{V}.\nabla p\,\delta V - p\nabla.\vec{V}\delta V - \rho g w\,\delta V + \rho\dot{q}\,\delta V$$

Where we have used $\vec{g} = g\hat{k}$

the second term on the left in above equation vanishes so that:

$$\rho\frac{de}{dt}+\rho\frac{d}{dt}\left(\frac{1}{2}\vec{V}.\vec{V}\right)=-\vec{V}.\nabla p-p\nabla.\vec{V}-\rho g w+\rho\dot{q}$$

This equation can be simplified by noting that if we take the dot product of V with the momentum equation we obtain (neglecting friction)

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

$$\vec{V}.\frac{d\vec{V}}{dt} = -2\vec{V}.(\vec{\Omega}\times\vec{V}) - \frac{1}{\rho}\vec{V}.\nabla p + \vec{V}.\vec{g} + \vec{V}.\vec{F}_r$$

$$\rho\frac{d}{dt}\left(\frac{1}{2}\vec{V}.\vec{V}\right) = -\vec{V}.\nabla p - \rho g w \ (1)$$

$$\vec{V}.\frac{d\vec{V}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{V}.\vec{V})$$

$$\vec{V}.\vec{g} = V.(-g\hat{k}) = -gw$$

The balance of mechanical energy owing to the motion of fluid element;

$$\rho\frac{de}{dt} = -p\nabla.V + \rho\dot{q} \ (2)$$

The thermal energy balance.

Using the definition of geopotential we have:

$$gw = g\frac{dz}{dt} = \frac{d}{dt}(gz) = \frac{d\Phi}{dt}$$

$$\rho\frac{d}{dt}\left(\frac{1}{2}\vec{V}.\vec{V} + \Phi\right) = -\vec{V}.\nabla p$$

The mechanical energy equation

The sum of the kinetic energy plus the gravitational energy is called the mechanical energy.

Thus states that following the motion, the rate of change of mechanical energy per unit volume equals the rate at which work is done by the pressure gradient force.

The thermal energy equation can be written in more familiar form by noting from

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla.\vec{V} = 0$$

$$\rho = 1/\alpha \qquad \rightarrow \qquad -\frac{1}{\rho^2} d\rho / dt = d\alpha / dt$$

$$\frac{1}{\rho}\nabla.\vec{V} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{d\alpha}{dt}$$

For dry air the internal energy per unit mass is given by

$$e = c_v T \qquad\qquad c_v = 717\ \mathrm{J\ kg^{-1}} K^{-1}$$

The thermal energy equation can be written in more familiar form

$$\rho \frac{de}{dt} = -p\nabla.V + \rho\dot{q} \quad (2)$$

$$e = c_v T$$

$$\frac{1}{\rho}\nabla.\vec{V} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{d\alpha}{dt}$$

$$c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} = \dot{q}$$

Which is the usual form of the thermodynamic energy equation.

Thus the first law of thermodynamics indeed is applicable to a fluid in motion.

Thermodynamics of the Dry Atmosphere

Taking the total derivative of the equation of state we obtain: $p\alpha = RT$

$$p\frac{d\alpha}{dt} + \alpha\frac{dp}{dt} = R\frac{dT}{dt}$$

Substituting for pda/dt in energy equation and using $c_p = c_v + R$, where

$c_p = 1004 \text{ J kg}^{-1}\text{K}^{-1}$

We can rewrite the first law of thermodynamics as:

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = \dot{q}$$

$$c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{q}$$

Dividing through by *T* and again using the equation of state, we obtain the entropy form of the first law of thermodynamics:

$$c_p \frac{d \ln T}{dt} - R \frac{d \ln p}{dt} = \frac{\dot{q}}{T} \equiv \frac{ds}{dt}$$