Geophysical Fluid Dynamics

Lecture 13

Sahraei

Department of Physics,

Razi University

https://sci.razi.ac.ir/~sahraei

The Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0$$

An alternative form of the continuity equation is obtained by applying the vector identity:

$$\nabla.(\rho \vec{V}) \equiv \rho \nabla.\vec{V} + \vec{V}.\nabla \rho$$

$$\rho \nabla.\vec{V} + \vec{V}.\nabla \rho = -\frac{\partial \rho}{\partial t}$$

$$\rho \nabla.\vec{V} = -\frac{\partial \rho}{\partial t} - \vec{V}.\nabla \rho$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}.\nabla$$

$$\frac{1}{\rho}\frac{d\rho}{dt}+\nabla.\vec{V}=0$$

The velocity divergence form of the continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt}=-\nabla.\vec{V}$$

آهنگ تغییر نسبی چگالی که با همگرایی سرعت برابر است.

$$\frac{d\rho}{dt}=-\rho\nabla.\vec{V}$$

آهنگ تغییرات تام چگالی

b) Lagrangian approach

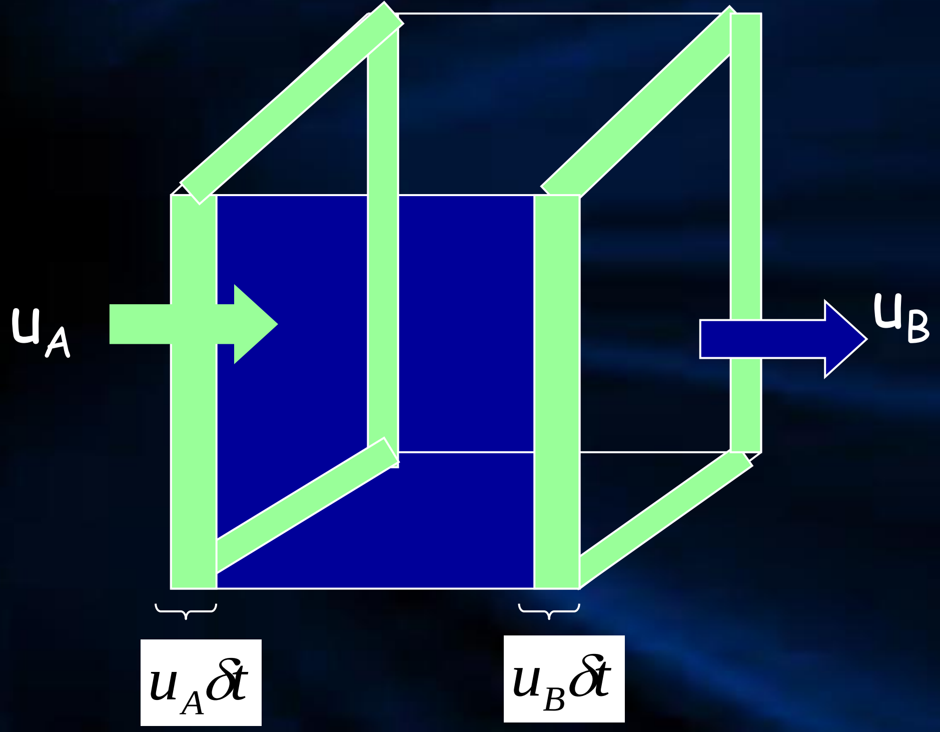
The physical meaning of divergence can be illustrated by the following method.

$$\delta V = \delta x \delta y \delta z$$

$$\delta M = \rho \delta V = \rho \delta x \delta y \delta z$$

$$\frac{d}{dt}(\delta M) = \rho \frac{d}{dt}(\delta V) + \delta V \frac{d\rho}{dt} = 0$$

$$\frac{1}{\delta M}\frac{d}{dt}(\delta M) = \frac{1}{\delta V}\frac{d}{dt}(\delta V) + \frac{1}{\rho}\frac{d\rho}{dt} = 0$$



$$\frac{1}{\delta V}\frac{d}{dt}(\delta V)=\frac{1}{\delta x}\frac{d}{dt}(\delta x)+\frac{1}{\delta y}\frac{d}{dt}(\delta y)+\frac{1}{\delta z}\frac{d}{dt}(\delta z)$$

δx ,δy ,δz =?

$$u_A=\frac{dx}{dt} \qquad u_B=\frac{d(x+\delta x)}{dt}$$

$$\delta u=u_B-u_A=\frac{d(x+\delta x)}{dt}-\frac{dx}{dt}=\frac{d(\delta x)}{dt}$$

$$\delta v=\frac{d(\delta y)}{dt} \qquad \delta w=\frac{d(\delta z)}{dt}$$

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt}=\frac{1}{\delta x}\delta u+\frac{1}{\delta y}\delta v+\frac{1}{\delta z}\delta w$$

$$\lim_{\delta x \delta y \delta z \to 0}\left[\frac{1}{\delta V}\frac{d}{dt}(\delta V)\right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla.\vec{V}$$

تغییرات نسبی حجم بسته هوا در واحد زمان برابر است با واگرایی سرعت

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla.\vec{V} = 0$$

آهنگ تغییرات نسبی چگالی برابر است با همگرایی سرعت

Continuity equation can be written in one of the two forms:

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{V})=0$$

شکل واگرایی جرم از معادله پیوستگی

آهنگ تغییر محلی چگالی با منفی واگرایی جرم برابر است.

The velocity divergence form of the continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt}+\nabla\cdot\vec{V}=0$$

آهنگ تغییر نسبی چگالی که با همگرایی سرعت برابر است.

special case

incompressible flow

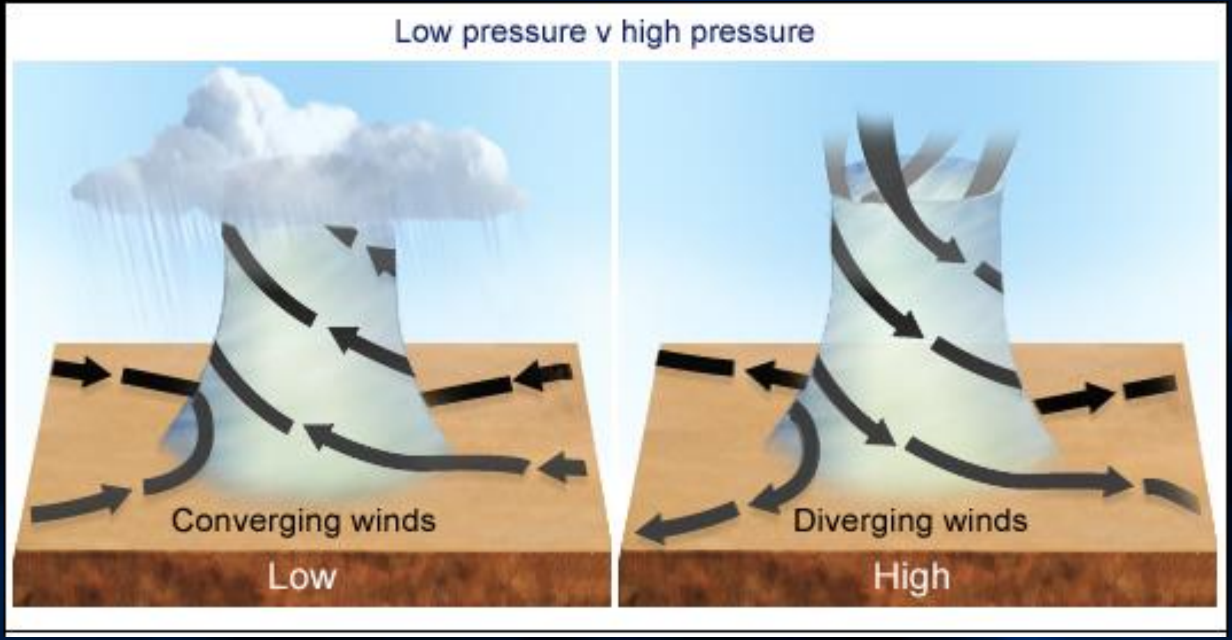
$$\frac{1}{\rho}\frac{d\rho}{dt}+\nabla\cdot\vec{V}=0$$

اگر شارش را غیر قابل تراکم در نظر بگیریم بنابراین چگالی ثابت خواهد بود.

$$\rho = cte \rightarrow \nabla.\vec{V}=0$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial w}{\partial z}=-\nabla_H.\vec{V}$$



Scale Analysis of Continuity Equation

When scaling the continuity equation, it is important to recognize that large variations in density are only relevant in the vertical.

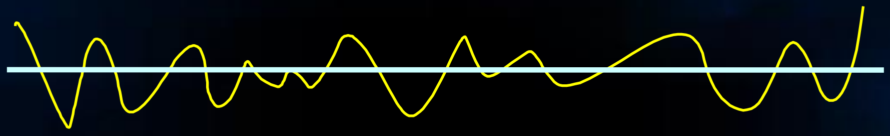
So we represent the density field as the sum of a horizontal reference value and a deviation from that value:

Substitute into velocity divergence form of the continuity equation:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0$$

Next, we will assume that the density field may be expressed as a sum of a referenced density which is function only of height and a small perturbation, that is:

$$\rho = \rho_0(z) + \rho'(x, y, z, t)$$



$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0 \qquad \rho\nabla.\vec{V} = -\frac{\partial \rho}{\partial t} - \vec{V}.\nabla\rho$$

$$\frac{1}{\rho_0}\left[\frac{\partial}{\partial t}(\rho_0 + \rho') + \vec{V}\cdot\nabla(\rho_0 + \rho')\right] + \nabla\cdot\vec{V} = 0$$

$$(\rho_0 + \rho')^{-1} = \rho_0^{-1}(1 + \rho'/\rho_0)^{-1} \cong \rho_0^{-1}(1 - \rho'/\rho_0) \qquad |\rho'/\rho_0| \ll 1$$

$$\frac{1}{\rho_0}\left[\left(\frac{\partial \rho_0}{\partial t}+\frac{\partial \rho'}{\partial t}\right)+u\left(\frac{\partial \rho_0}{\partial x}+\frac{\partial \rho'}{\partial x}\right)+v\left(\frac{\partial \rho_0}{\partial y}+\frac{\partial \rho'}{\partial y}\right)+w\left(\frac{\partial \rho_0}{\partial z}+\frac{\partial \rho'}{\partial z}\right)\right]+\nabla\cdot\vec{V}=0$$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t}+u\frac{\partial \rho'}{\partial x}+v\frac{\partial \rho'}{\partial y}+w\frac{d\rho_0}{dz}+w\frac{\partial \rho'}{\partial z}\right]+\nabla\cdot\vec{V}=0$$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t}+\vec{V}\cdot\nabla\rho'\right]+\frac{w}{\rho_0}\frac{d\rho_0}{dz}+\nabla\cdot\vec{V}=0$$

A **B** **C**

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \vec{V}\cdot\nabla\rho'\right] + \frac{w}{\rho_0}\frac{d\rho_0}{dz} + \nabla\cdot\vec{V} = 0$$

A **B** **C**

A: $\frac{\rho'}{\rho_0}\frac{U}{L} \approx 10^{-7}\,s^{-1}$

$\frac{\rho'}{\rho_0} \approx 10^{-2}$ For synoptic-scale motions

B: $\frac{W}{H} \approx 10^{-6}\,s^{-1}$

C: $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx 10^{-1}\frac{U}{L} \approx 10^{-6}\,s^{-1}$

Thus, terms B and C are each an order of magnitude greater than term A, and, to a first approximation, terms B and C balance in the continuity equation. To a good approximation, then:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+w\frac{d}{dz}(\ln\rho_0)=0$$

If only the first three terms in the above equation were present, then the atmosphere would be incompressible.

or, alternatively, in vector form $\nabla.(\rho_0\vec{V})=0$

Thus, for synoptic-scale motions the mass flux computed using the basic state density ρ_0 is nondivergent. This approximation is similar to the idealization of incompressibility, which is often used in fluid mechanics. However, an incompressible fluid has a density constant following the motion

$$\frac{d\rho}{dt}=0 \qquad \frac{1}{\rho}\frac{d\rho}{dt}+\nabla\cdot\vec{V}=0 \qquad \nabla.\vec{V}=0$$