Geophysical Fluid Dynamics

Lecture 12

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The Hydrostatic Approximation

A similar scale analysis can be applied to the vertical Component of the momentum equation.

$$z-eq. \quad \frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u \cos\phi + g + F_{rz}$$

$Scales$	UW/L	$\frac{U^2}{a}$	$\frac{P_0}{\rho H}$	$f_0 U$	g	$\frac{\nu W}{H^2}$
(ms^{-2})	10^{-7}	10^{-5}	10	10^{-3}	10	10^{-15}

The biggest terms give the hydrostatic approximation:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g$$

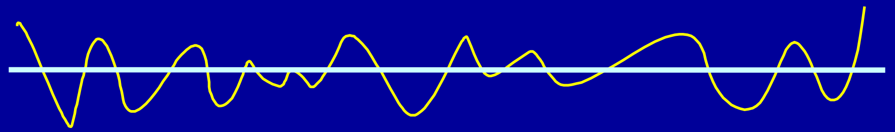
$P_0(z)$ is the horizontally averaged pressure at each height.

$\rho_0(z)$ is standard density.

We may then write the total pressure and density fields as:

$$P(x,y,z,t) = P_0(z) + P'(x,y,z,t)$$

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$



Where P′ and ρ′ are deviation from the standard values of pressure and density (perturbation).

For an atmosphere at rest, would thus be zero.

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(P_0 + P') - g$$

$$(\rho_0 + \rho')^{-1} = \rho_0^{-1}(1 + \rho'/\rho_0)^{-1} \cong \rho_0^{-1}(1 - \rho'/\rho_0)$$

$$\rho_0^{-1}(1 - \rho'/\rho_0)(\partial P_0/\partial z + \partial P'/\partial z) + g =$$

$$\rho_0^{-1}\left(\frac{\partial P_0}{\partial z} + \frac{\partial P'}{\partial z} - \frac{\rho'}{\rho_0}\frac{\partial P_0}{\partial z} - \frac{\rho'}{\rho_0}\frac{\partial P'}{\partial z}\right) + g$$

$$-g \qquad \approx \frac{\delta P}{\rho_0 H} \approx 10^{-1} ms^{-2} \qquad (\rho'/\rho_0)g = 10^{-1} \qquad 10^{-3}$$

$$\frac{1}{\rho_0}\frac{\partial P'}{\partial z} + \frac{\rho'}{\rho_0} g = 0 \qquad \frac{1}{\rho_0}\left(\rho' g + \frac{\partial P'}{\partial z}\right) = 0 \qquad \frac{1}{\rho'}\frac{\partial P'}{\partial z} = -g$$

ABOVE THE BOUNDARY LAYER

$$\frac{du}{dt}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+K\frac{\partial^2 u}{\partial z^2}+2\Omega v\sin\varphi-2\Omega w\cos\varphi$$

$$\frac{dv}{dt}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+K\frac{\partial^2 v}{\partial z^2}-2\Omega u\sin\varphi$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{a}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+0-g+2\Omega u\cos\varphi$$

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+fv \qquad \frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial P}{\partial y}-fu \qquad 0=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g$$

WITHIN THE BOUNDARY LAYER

$$\frac{du}{dt} - \cancel{\frac{uv\tan\varphi}{a}} + \cancel{\frac{uw}{a}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} \qquad + 2\Omega v\sin\varphi - \cancel{2\Omega w\cos\varphi}$$

$$\frac{dv}{dt} + \cancel{\frac{u^2\tan\varphi}{a}} + \cancel{\frac{vw}{a}} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} \qquad - 2\Omega u\sin\varphi$$

$$\cancel{\frac{dw}{dt}} - \cancel{\frac{u^2+v^2}{a}} \qquad = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 0 - g + \cancel{2\Omega u\cos\varphi}$$

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + fv \qquad \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - fu \qquad 0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g$$

The Hidden Simplicity of Atmospheric Dynamics:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu$$

the boundary layer, all horizontal parcel accelerations can be understood by comparing the magnitude and direction of the pressure gradiant and Coriolis forces

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - fu$$

the boundary layer, all horizontal parcel accelerations can be understood by comparing the magnitude and direction of the pressure gradient, Coriolis and frictional forces

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g$$

The atmosphere is in hydrostatic balance-vertical PGF balances gravity on synoptic scales

The Continuity Equation

We turn now to the second of the three fundamental conservation principles, conservation of mass.

The mathematical relationship that expresses conservation of mass for a fluid is called the continuity equation.

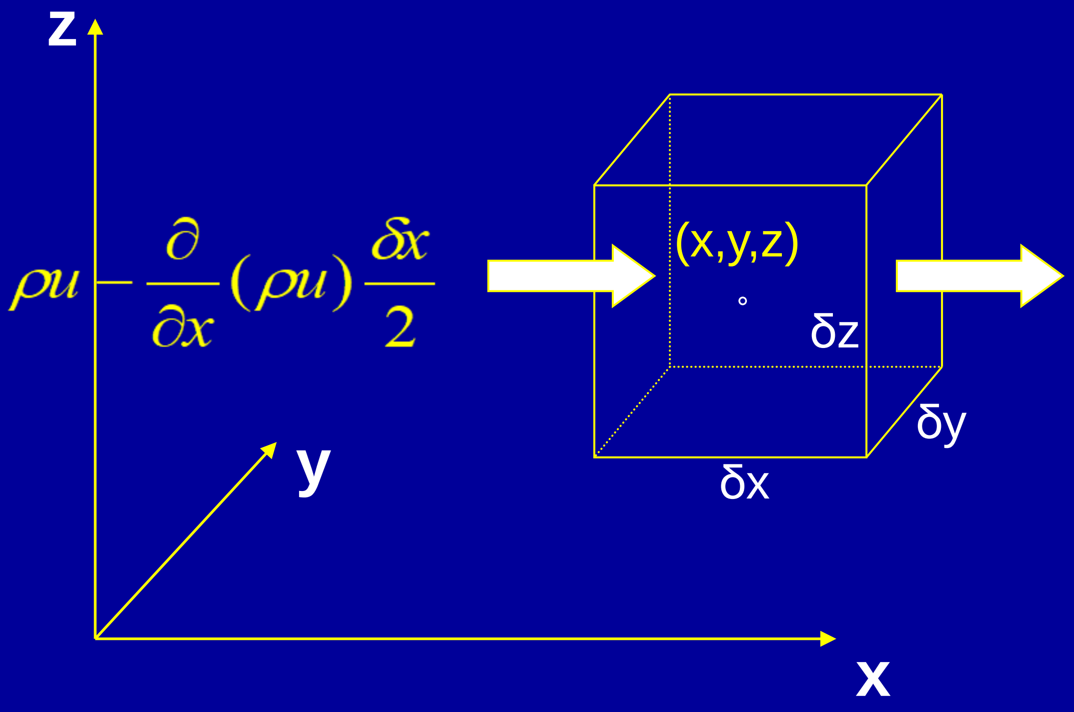
a) Eulerian approach

b) Lagrangian approach

a) Eulerian approach

We consider a volume element δx δy δz that is fixed in a Cartesian coordinate frame as shown in Fig.

$$\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$$



$$\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$$

$$\left[\ \rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\ \right]$$

$$\left[\ \rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\ \right]$$

Since the area of these faces is δy δz, the net rate of flow into the volume owing to the x velocity component is:

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z \;-\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z$$

$$= -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z$$

$$-\frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z$$

$$= -\frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z$$

Thus, the net rate of mass inflow is

$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$$

$$=-\nabla.(\rho\vec{V})\delta V$$

The mass inflow per unit volume is just

$$-\nabla.(\rho\vec{V})$$

which must equal the rate of mass increase per unit volume.

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$t = t_0$ $\delta m = \rho\delta V$

$$\frac{\partial(\delta m)}{\partial t} = \frac{\partial \rho}{\partial t}\delta V \qquad \frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{V}) = 0$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0$$

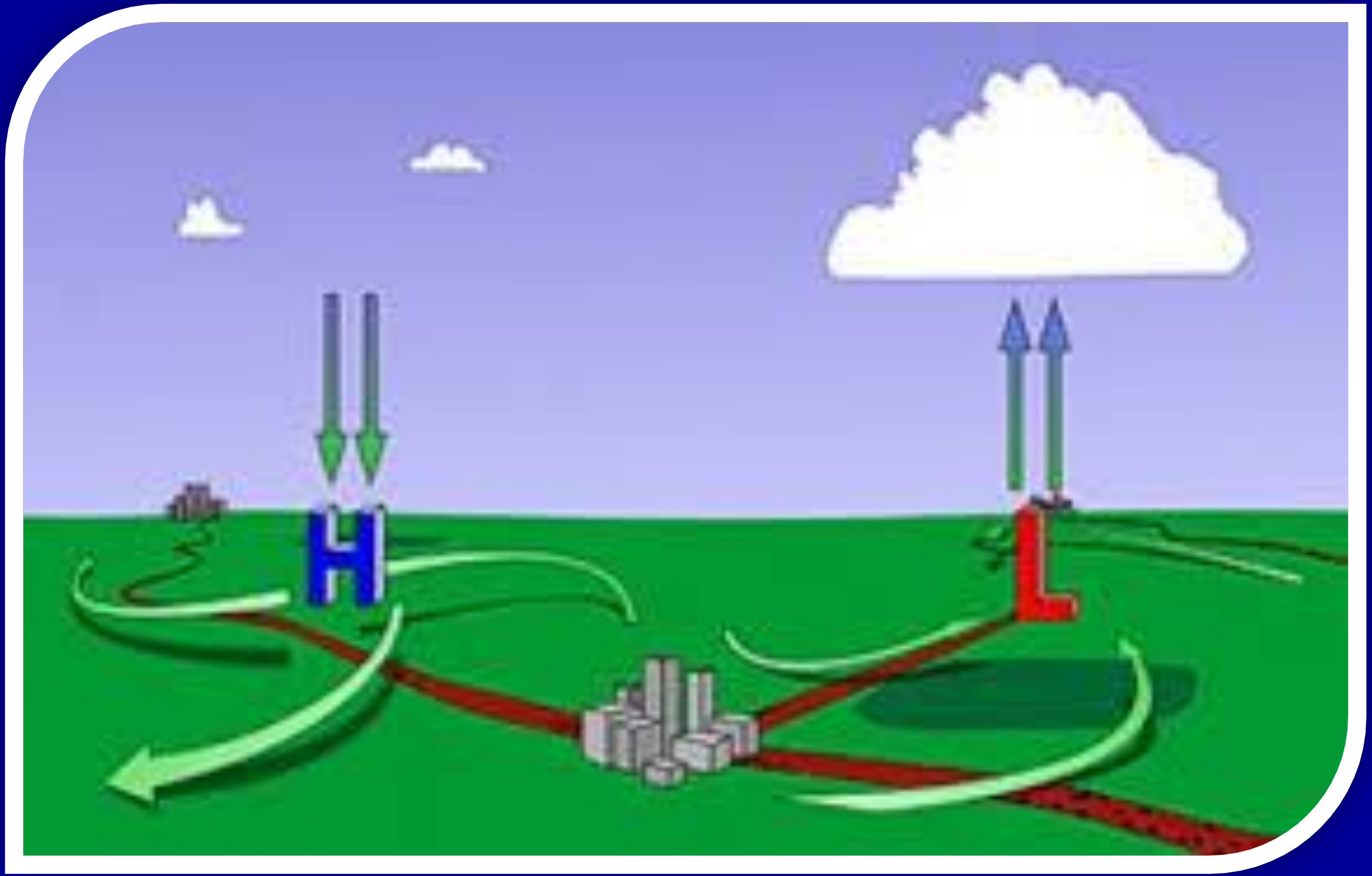
A physical interpretation of this form of the continuity equation is that the change in density at a fixed point in space is dependent upon the divergence of the mass flux.

If there is divergence of the mass flux then

$\nabla.(\rho \vec{V}) > 0$ and density will decrease $\frac{\partial \rho}{\partial t} < 0$

If there is convergence of the mass flux then

$\nabla.(\rho \vec{V}) < 0$ and density will increase $\frac{\partial \rho}{\partial t} > 0$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