Geophysical Fluid Dynamics

Lecture 10

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Vector momentum equation in rotating coordinates

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

unit vectors are a function of position on the earth (not Cartesian)

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

The (x,y,z) coordinates system defined in this way is not a Cartesian coordinates system , because the directions of the unit vectors depend on their position on the earth's surface.

This position dependence of the unit vectors must be taken into account when the acceleration vector is expended into its components on the sphere.

We first consider $d\hat{i}/dt$

$$\frac{d\hat{i}}{dt} = \cancel{\frac{\partial \hat{i}}{\partial t}} + u\frac{\partial \hat{i}}{\partial x} + v\cancel{\frac{\partial \hat{i}}{\partial y}} + w\cancel{\frac{\partial \hat{i}}{\partial z}}$$

At a point (constant x,y,z) none of the unit vectors change with time so

$$\frac{\partial \hat{i}}{\partial t} = 0$$

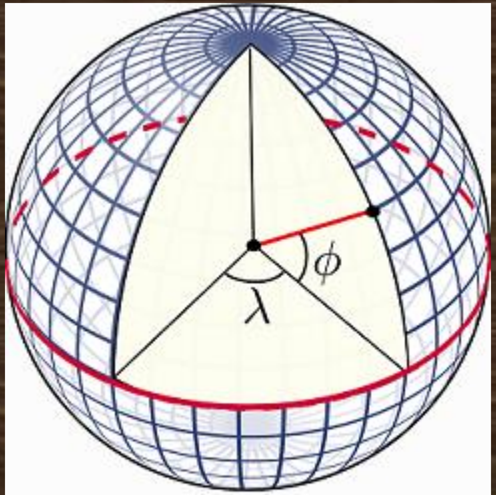
As one moves north or south the $\hat{i}$ direction experiences no change so

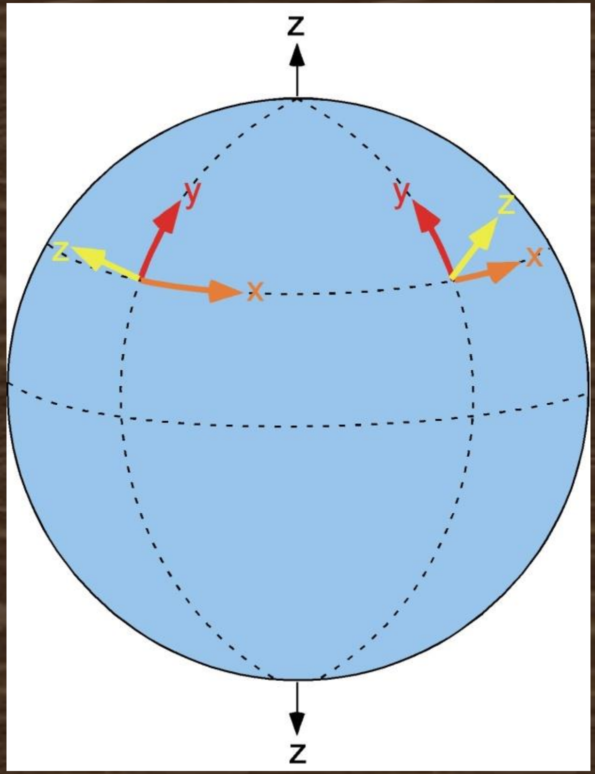
$$\frac{\partial \hat{i}}{\partial y} = 0$$

As one moves up or down the $\hat{i}$ direction experiences no change so

$$\frac{\partial \hat{i}}{\partial z} = 0$$

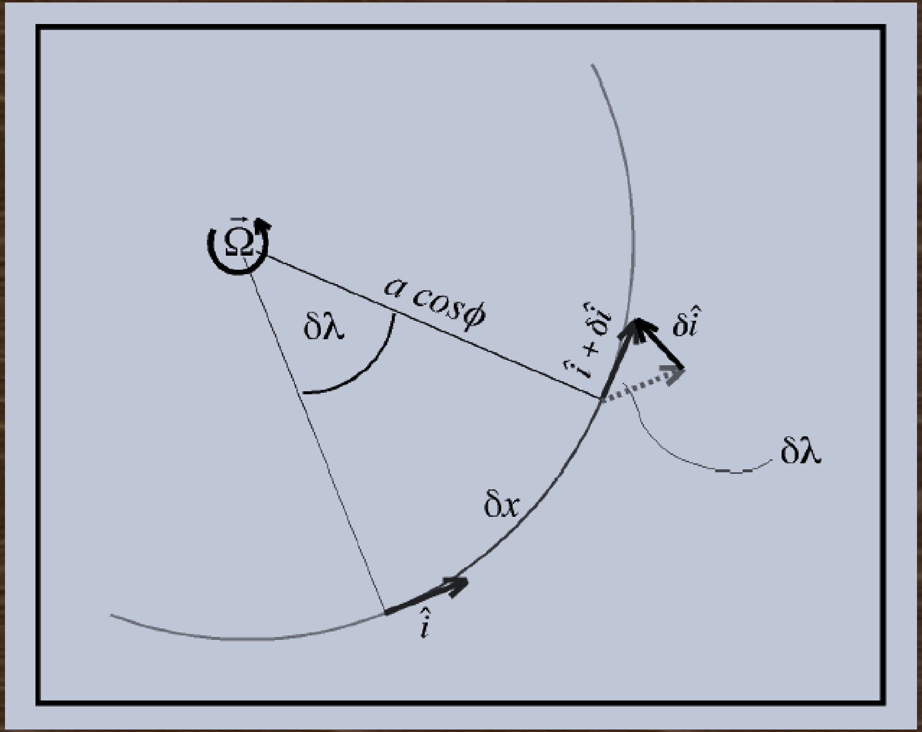
So: $\frac{d\hat{i}}{dt} = u\frac{\partial \hat{i}}{\partial x}$ $\quad\hat{i}$ is a function only of x





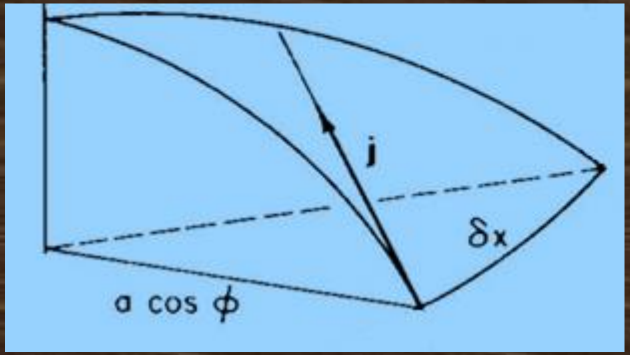
$$\frac{d\hat{i}}{dt} = u\frac{\partial\hat{i}}{\partial x}$$

From figure on right looking down at north pole at latitude ϕ:



$$\left|\delta\hat{i}\right| = \left|\hat{i}\right|\delta\lambda$$

$$\delta x = a\cos\varphi\,\delta\lambda$$

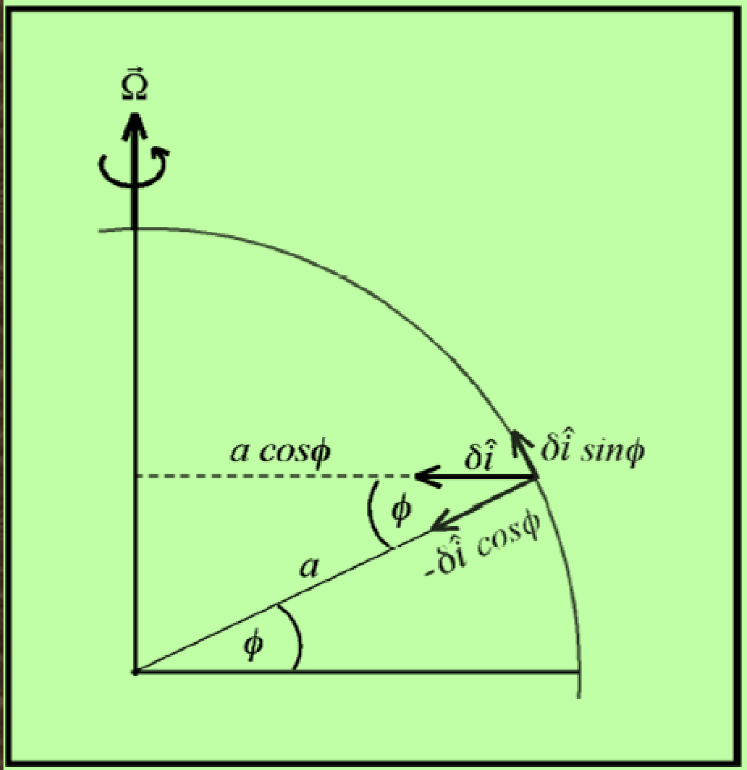


$$\left|\frac{\delta\hat{i}}{\delta x}\right| = \frac{1}{a\cos\varphi}$$

This gives us the magnitude, but not the direction.

Note that is $\partial\hat{i}$ pointed toward the center of the earth at the original point

We see that $\delta\hat{i}$ has components in the $\hat{j}$ and $-\hat{k}$ directions



The unit vector describing the direction of $\delta\hat{i}$ has two components:

$$\delta\hat{i} = \hat{j}\sin\varphi - \hat{k}\cos\varphi$$

So:

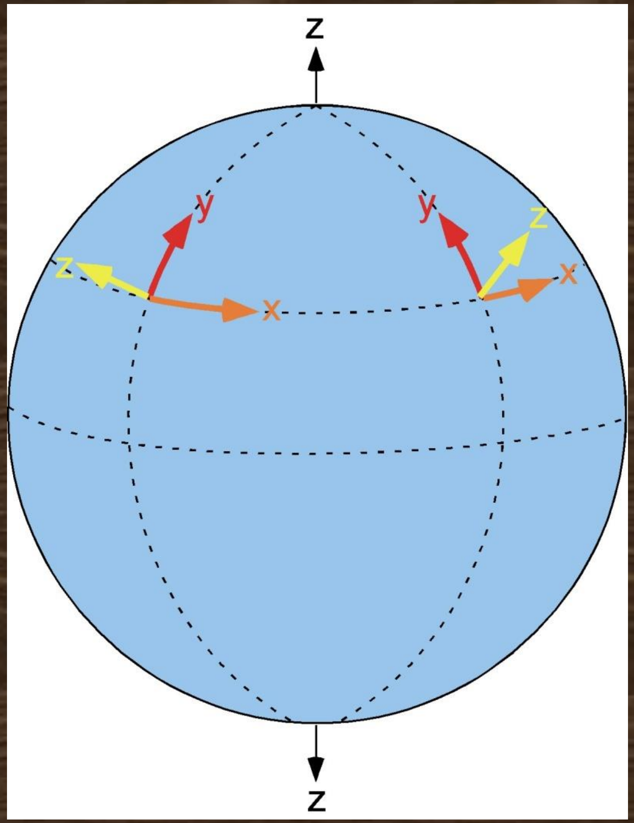
$$u\frac{d\hat{i}}{dt} = u\frac{\partial\hat{i}}{\partial x} = \frac{u(\hat{j}\sin\varphi - \hat{k}\cos\varphi)}{a\cos\varphi} = \hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}$$

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

We still need to determine what these are

$$\frac{d\hat{j}}{dt} = \cancel{\frac{\partial\hat{j}}{\partial t}} + u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y} + \cancel{w\frac{\partial\hat{j}}{\partial z}}$$



$\hat{j}$ does not change with time or elevation, but does change in the x and y directions

$$\frac{d\hat{j}}{dt} = u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y}$$

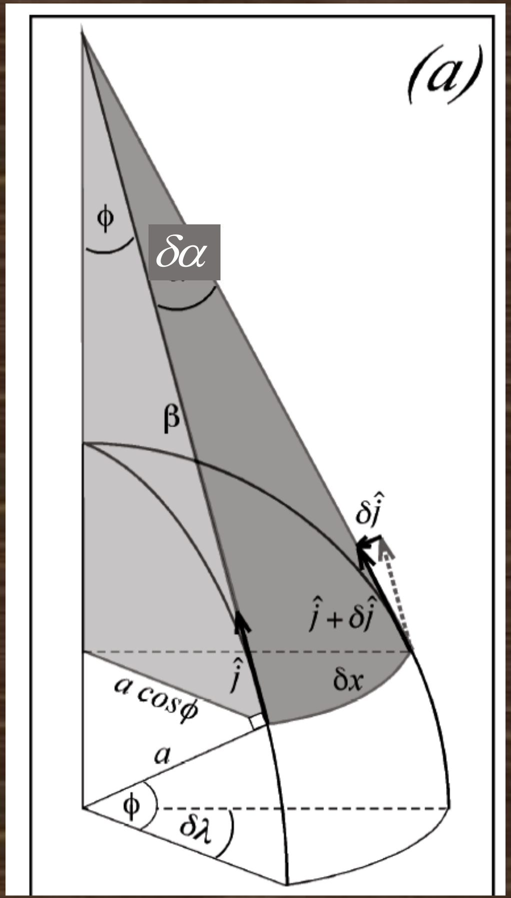
Lets first figure out $\frac{\partial \hat{j}}{\partial x}$

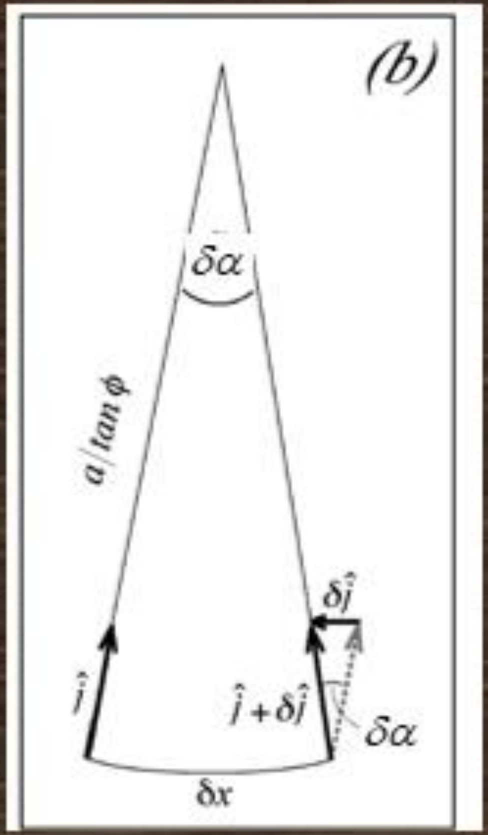
Look at light gray triangle in (a):

$$\delta x = \beta \delta \alpha \qquad a\cos\varphi = \beta \sin\varphi$$

or $$\beta = \frac{a}{\tan\varphi}$$

Dark gray triangle is shown in a different view in (b)





$$\delta x = \frac{a}{\tan\varphi}\delta\alpha$$

$$\delta\hat{j} = \hat{j}\delta\alpha$$

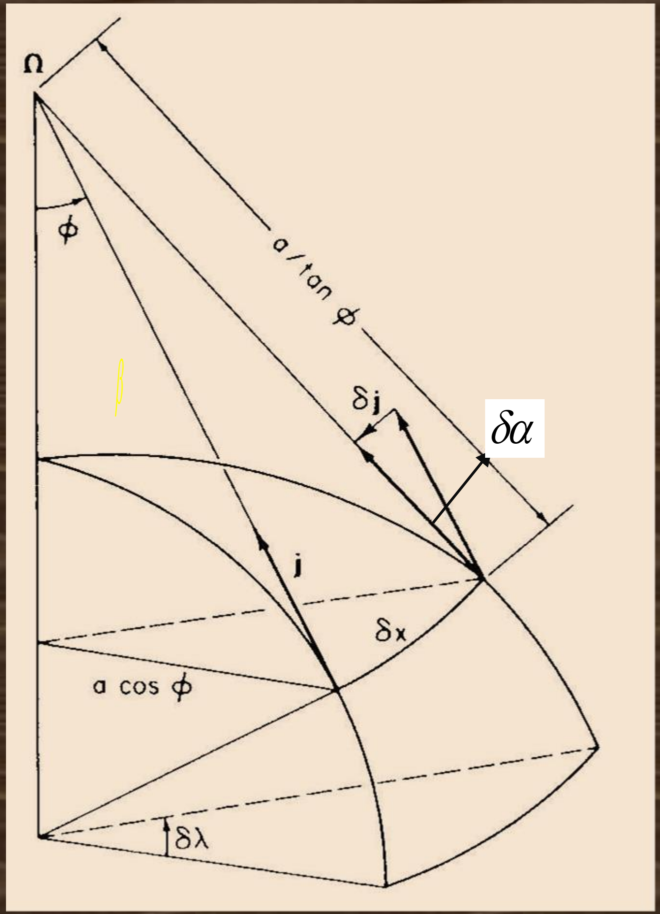
So: $$\left|\frac{\delta\hat{j}}{\delta x}\right| = \frac{\tan\varphi}{a}$$

Direction of is the -x direction $\delta\hat{j}$

Since the vector $\partial\hat{j} / \partial x$ is directed in the negative x direction,

$$\lim_{\delta x \to 0} \frac{\delta\hat{j}}{\delta x} = \frac{\partial\hat{j}}{\partial x} = -\frac{tg\phi}{a}\hat{i}$$

$$\frac{d\hat{j}}{dt} = u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y} = -u\frac{tg\varphi}{a}\hat{i} + v\frac{\partial\hat{j}}{\partial y}$$



$$v\frac{\partial \hat{j}}{\partial y}$$

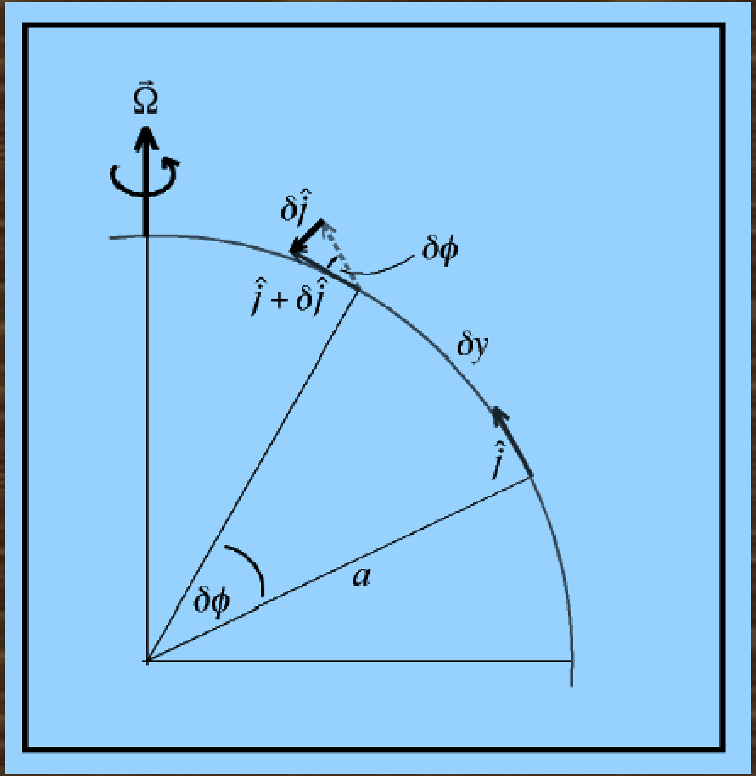
From Fig. it is clear that for northward motion

$$\left|\delta\hat{j}\right| = \delta\varphi$$

$$\delta y = a\,\delta\phi$$

$$\lim_{\delta y\to 0}\frac{\delta\hat{j}}{\delta y} = \frac{\partial\hat{j}}{\partial y} = -\frac{1}{a}\hat{k}$$

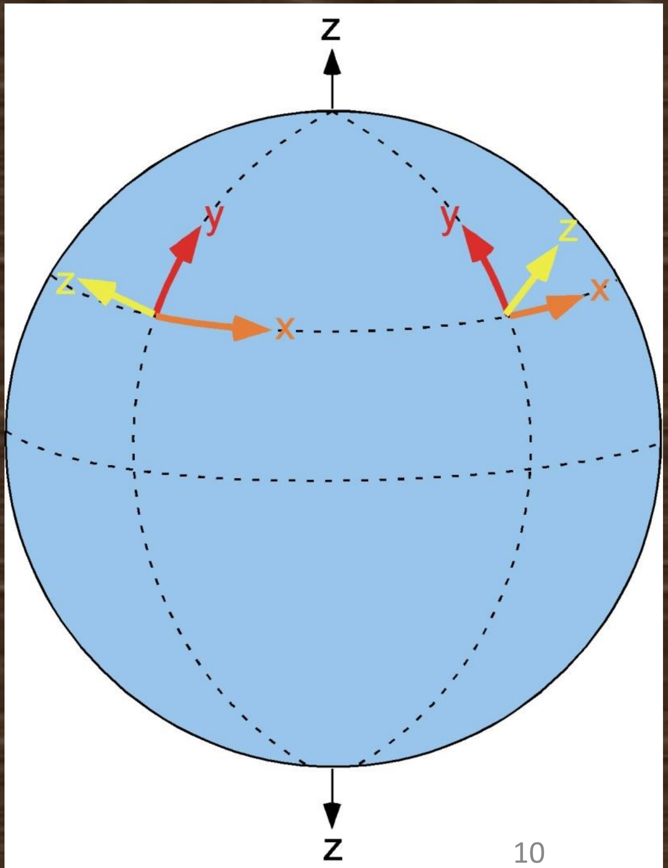
$$\frac{d\hat{j}}{dt} = u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y} \qquad \frac{d\hat{j}}{dt} = -\frac{u\,tg\,\phi}{a}\hat{i} - \frac{v}{a}\hat{k}$$



$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{V}_i}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\left(-\hat{i}\frac{u\tan\varphi}{a} - \hat{k}\frac{v}{a}\right) + w\frac{d\hat{k}}{dt}$$

$$\frac{d\hat{k}}{dt} = \cancel{\frac{\partial\hat{k}}{\partial t}} + u\frac{\partial\hat{k}}{\partial x} + v\frac{\partial\hat{k}}{\partial y} + \cancel{w\frac{\partial\hat{k}}{\partial z}}$$



$\hat{k}$ does not change with time or elevation, but does change in the x and y directions

$$\frac{d\hat{k}}{dt} = u\frac{\partial\hat{k}}{\partial x} + v\frac{\partial\hat{k}}{\partial y}$$

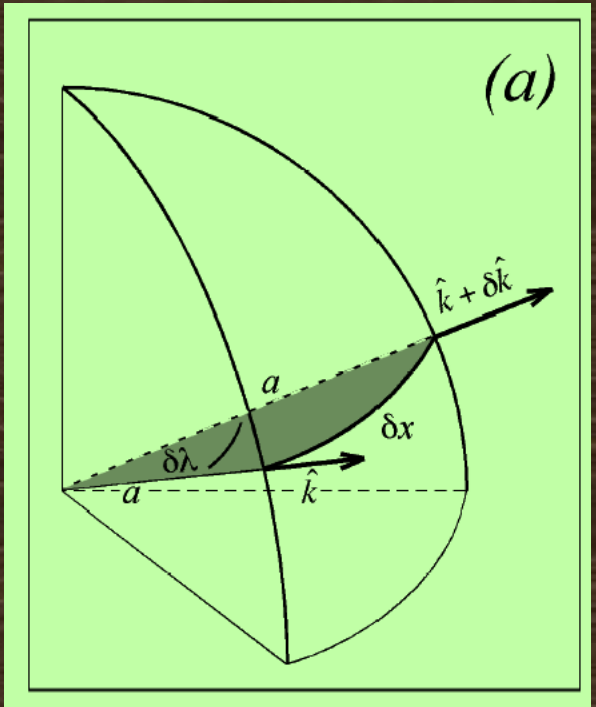
$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}$$

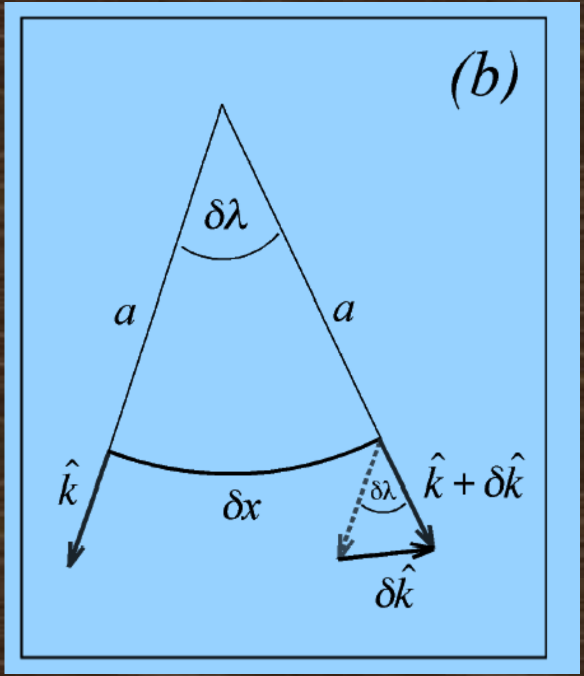
Let's do $\frac{\partial \hat{k}}{\partial x}$ first

$$\delta x = a\delta\lambda \qquad \left|\delta\hat{k}\right| = \left|\hat{k}\delta\lambda\right| = \delta\lambda$$

Direction of $\delta\hat{k}$ is the positive $\hat{i}$ direction

$$\frac{\partial \hat{k}}{\partial x} = \frac{1}{a}\hat{i} \qquad u\frac{\partial \hat{k}}{\partial x} = \hat{i}\frac{u}{a}$$

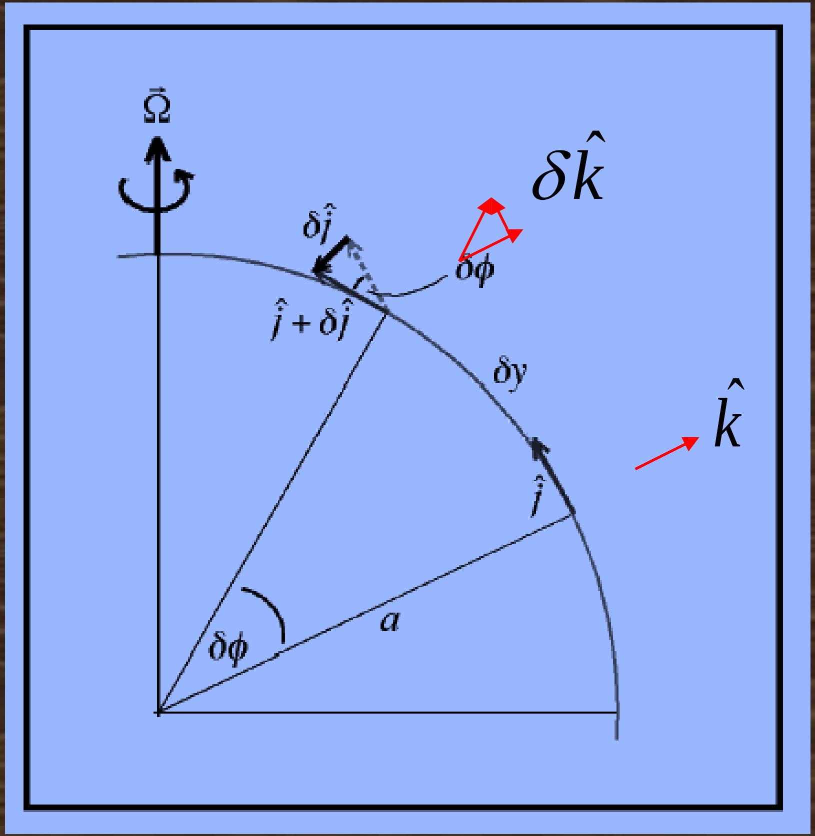




$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}$$

Let's do $\frac{\partial \hat{k}}{\partial y}$ next

$$\delta y = a\delta\varphi \qquad \left|\delta\hat{k}\right| = \left|\hat{k}\delta\varphi\right| = \delta\varphi$$



Direction of $\delta\hat{k}$ is the positive $\hat{j}$ direction

$$\frac{\partial \hat{k}}{\partial y} = \frac{1}{a}\hat{j} \qquad \frac{d\hat{k}}{dt} = \frac{u}{a}\hat{i} + \frac{v}{a}\hat{j}$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\left(-\hat{i}\frac{u\tan\varphi}{a} - \hat{k}\frac{v}{a}\right) + w\left(\frac{u}{a}\hat{i} + \frac{v}{a}\hat{j}\right)$$

or

$$\frac{d\vec{V}}{dt} = \hat{i}\left(\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a}\right) + \hat{j}\left(\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a}\right) + \hat{k}\left(\frac{dw}{dt} - \frac{u^2 - v^2}{a}\right)$$

A Complication of Spherical Coordinates

Newton's second law in an inertial coordinate system

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Newton's second law in a spherical coordinate system

$$m\left[\frac{d\vec{V}}{dt} + \hat{i}\left(-\frac{uv\tan\varphi}{a} + \frac{uw}{a}\right) + \hat{j}\left(\frac{u^2\tan\varphi}{a} + \frac{vw}{a}\right) + \hat{k}\left(-\frac{u^2 - v^2}{a}\right)\right] = \sum Forces$$

Now let's look at the right side of the equation!

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Vector momentum equation in rotating coordinates

Total derivative

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = \ldots$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

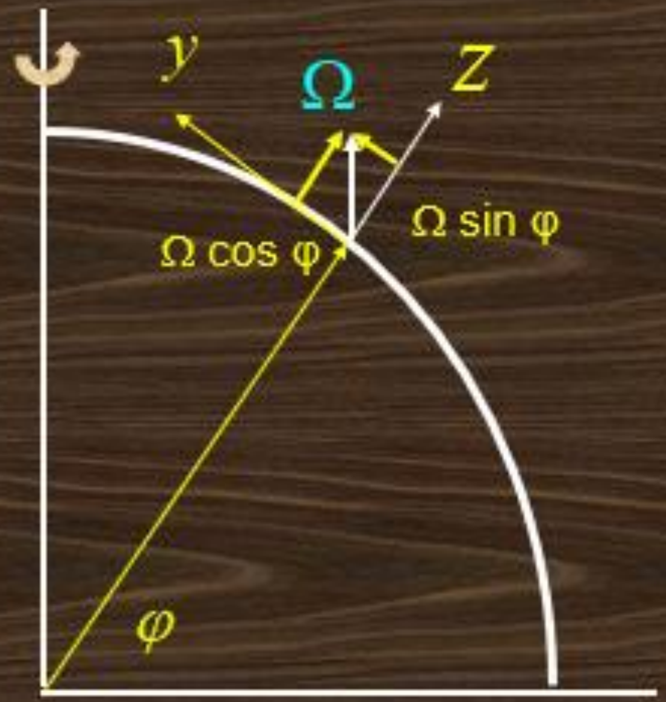
Coriolis acceleration

$$-2\vec{\Omega}\times\vec{V} = -2\Omega\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \cos\varphi & \sin\varphi \\ u & v & w \end{vmatrix} = (2\Omega v\sin\varphi - 2\Omega w\cos\varphi)\hat{i} - 2\Omega u\sin\varphi\hat{j} + 2\Omega\cos\varphi\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\varphi + \ldots$$



$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Pressure gradient term

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Gravity

$\vec{g} = -g\hat{k}$ $\quad$ g is a positive scalar = 9.8 m s^{-2} at earth's surface

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Friction

$$\vec{F}_r = F_{rx}\hat{i} + F_{ry}\hat{j} + F_{rz}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz}$$

Momentum Equations in Spherical Coordinates

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + F_{rz}$$

The terms proportional to *1/a* on the left hand sides are called the curvature terms; they arise owing to the curvature of the earth.

Because they are nonlinear (that is, they are quadratic in the dependent variables) they are difficult to handle in theoretical analyses.