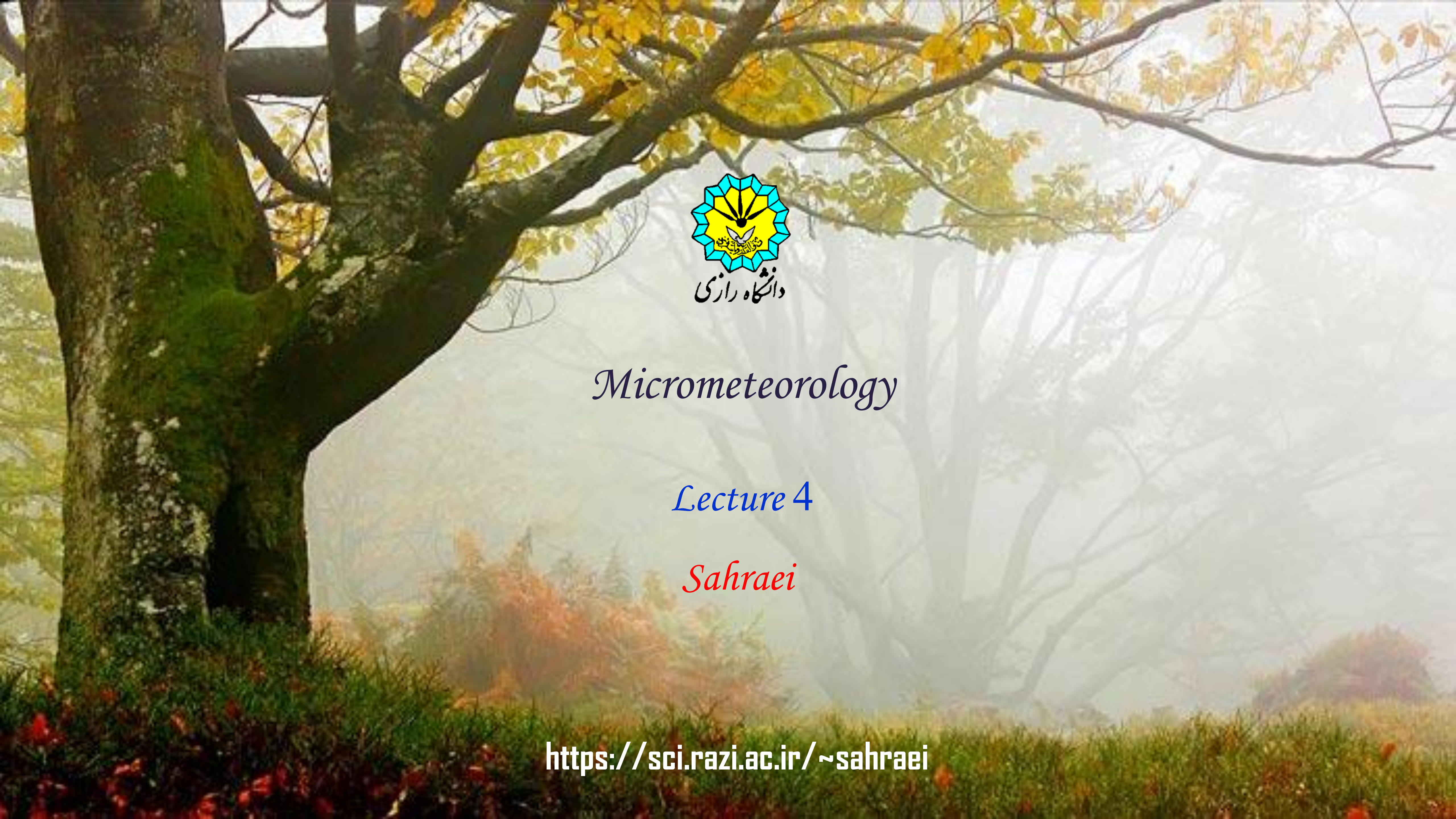


دانشگاه رازی

Micrometeorology

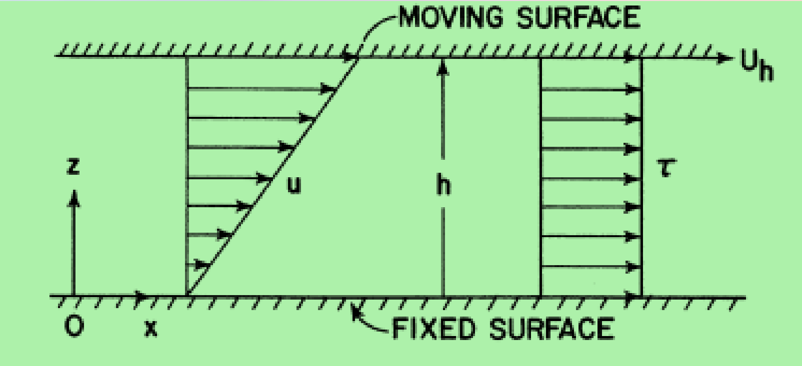
Lecture 4

Sahraei

https://sci.razi.ac.ir/~sahraei

Momentum Fluxes

$$\tau = \mu \frac{\partial u}{\partial z} - \rho \overline{u'w'}$$



$$\tau = \mu(\partial u/\partial z)$$

$$Momentum\ Flux = \rho\overline{uw}$$

$$u = \bar{u} + u'$$

$$w = \bar{w} + w'$$

$$\overline{uw} = \bar{u}\bar{w} + \overline{u'w'}$$

$$\bar{w} = 0$$

$$\overline{uw} \approx \overline{u'w'}$$

$$\tau = -\rho\overline{u'w'}$$

Momentum Fluxes

Magnitude of Reynolds stress at ground surface

$$\left|\tau_z\right| = \rho\left[\left(\overline{w'u'}\right)^2 + \left(\overline{w'v'}\right)^2\right]^{1/2}$$

Kinematic vertical turbulent momentum flux (m^2 s^{-2})

$$\overline{w'u'} = -\frac{\tau_{zx}}{\rho}$$

$$\overline{w'v'} = -\frac{\tau_{zy}}{\rho}$$

Friction wind speed (m s^{-1})

Scaling param. for surface-layer vert. flux of horiz. momentum

$$u_* = \left[\left(\overline{w'u'}\right)_s^2 + \left(\overline{w'v'}\right)_s^2\right]^{1/4} = \left(\left|\mathrm{t}_z\right|/\rho_a\right)_s^{1/2}$$

How do we measure u'?

$$u=(\bar{u}+u')$$

A sonic anemometer measures at very high sampling rates. This is typically at 10 or 20 Hz. Data from a high temporal resolution time series is used to calculate the mean of the time series and subsequently a perturbation from the mean.

Once the perturbations are calculated, turbulent statistics can be calculated.

$$u'=u-\bar{u}$$

The Mean

Time ( ), space ( ), and ensemble ( ) averages are three ways to define a mean. The time average applies at one specific point in space, and consists of a sum or integral over time period P. For any variable, $A(t, s)$, that is a function of time, t, and space, s:

$$ {}^{t}\overline{A}(s) = \frac{1}{N}\sum_{i=0}^{N-1} A(i, s) \qquad \text{or} \qquad {}^{t}\overline{A}(s) = \frac{1}{P}\int_{t=0}^{P} A(t, s)\,dt $$

where $t = i\,\Delta t$ for the discrete case.

The spatial average, which applies at some instant in time, is given by a sum or integral over spatial domain S:

$$ {}^{s}\overline{A}(t) = \frac{1}{N}\sum_{j=0}^{N-1} A(t,j) \qquad \text{or} \qquad {}^{s}\overline{A}(t) = \frac{1}{S}\int_{t=0}^{S} A(t,s)\,ds $$

where $s = j\Delta s$ in the discrete case.

An ensemble average consists of the sum over N identical experiments:

$$ {}^{e}\overline{A}(t,s) = \frac{1}{N}\sum_{i=0}^{N-1} A_i(t,s) $$

In the equations above, $\Delta t = P/N$ and $\Delta S = S/N$,

where N is the number of data points.

This is called the ergodic condition, which is often assumed to make the turbulence problem more tractable:

$$ {}^{e}\overline{(\ )} = {}^{t}\overline{(\ )} = {}^{s}\overline{(\ )} \equiv \overline{(\ )} $$

Rules of Averaging

Let A and B be two variables that are dependent on time, and let c represent a constant.

To find the average of the sum of A and B, we can employ the equations of the previous section with some basic rules of summation or integration to show that:

$$\overline{(A+B)} = \bar{A} + \bar{B}$$

In terms of discrete sums, the average is :

$$\overline{(A+B)} = \frac{1}{N}\sum_{i=0}^{N-1}(A_i + B_i) = \frac{1}{N}\left(\sum_i A_i + \sum_i B_i\right) = \frac{1}{N}\sum_i A_i + \frac{1}{N}\sum_i B_i = \bar{A} + \bar{B}$$

In terms of continuous integrals:

$$\overline{(A+B)} = \frac{1}{P}\int_{t=0}^{P}(A+B)\,dt = \frac{1}{P}\left(\int_t A\,dt + \int_t B\,dt\right) = \frac{1}{P}\int_t A\,dt + \frac{1}{P}\int_t B\,dt = \bar{A} + \bar{B}$$

Both the sum and integral approaches give the same answer, as expected

We can use similar methods to show that:

$$\overline{(c\,A)} = c\,\overline{(A)}$$

$$\bar{c} = c$$

An important consequence of averaging is that an average value acts like a constant when averaged a second time over the same time period, P:

Define

$$\frac{1}{P}\int_{t=0}^{P} A(t,s)\,dt \equiv \overline{A}(P,s)$$

Therefore

$$\frac{1}{P}\int_{t=0}^{P} \overline{A}(P,s)\,dt \equiv \overline{A}(P,s)\,\frac{1}{P}\int_{t=0}^{P} dt \equiv \overline{A}(P,s)$$

Leaving

$$\overline{(\overline{A})} = \overline{A}$$

Similarly, it can be shown that:

$$\overline{(\overline{A}\,B)} = \overline{A}\;\overline{B}$$

To summarize the rules of averaging:

$$\overline{c} = c$$

$$\overline{(c\,A)} = c\,\overline{A}$$

$$\overline{(\overline{A})} = \overline{A}$$

$$\overline{(\overline{A}\,B)} = \overline{A}\;\overline{B}$$

$$\overline{(A+B)} = \overline{A} + \overline{B}$$

$$\overline{\left(\frac{dA}{dt}\right)} = \frac{d\,\overline{A}}{dt}$$

Reynolds Averaging

The averaging rules of the last section can now be applied to variables that are split into mean and turbulent parts.

Let $A = \overline{A} + a'$ $\qquad B = \overline{B} + b'$

Starting with the instantaneous value, A, for example, we can find its mean using the fifth and third rules of the previous section:

$$\overline{(A)} = \overline{(\overline{A} + a')} = \overline{(\overline{A})} + \overline{a'} = \overline{A} + \overline{a'}$$

The only way that the left and right sides can be equal is if $\overline{a'} = 0$

This result is not surprising if one remembers the definition of a mean value. By definition, the sum of the positive deviations from the mean must equal the sum of the negative deviations. Thus the deviations balance when summed, as implied in the above average.

Another example: start with the product $\overline{B}\,a'$ and find its average. Employing the above result together with the fourth averaging rule, we find that

$$\overline{(\overline{B}\,a')} = \overline{B}\,\overline{a'} = \overline{B}\cdot 0 = 0$$

Similarly, $\overline{\overline{A}\,b'} = 0.$

The average of the product of *A* and B is

$$\overline{(A\cdot B)} = \overline{(\overline{A} + a')(\overline{B} + b')}$$

$$= \overline{(\overline{A}\,\overline{B} + a'\,\overline{B} + \overline{A}\,b' + a'\,b')}$$

$$= \overline{(\overline{A}\,\overline{B})} + \overline{(a'\,\overline{B})} + \overline{(\overline{A}\,b')} + \overline{(a'\,b')}$$

$$= \overline{A}\,\overline{B} + 0 + 0 + \overline{a'\,b'}$$

$$= \overline{A}\,\overline{B} + \overline{a'\,b'}$$

The nonlinear product $\overline{a'\,b'}$ is NOT necessarily zero.

The same conclusion holds for other nonlinear variables such as:

$$\overline{a'^2}\ ,\ \overline{a'b'^2}\ ,\ \overline{a'^2\,b'^2}$$

In fact, these nonlinear terms must be retained to properly model turbulence.

This is a dramatic difference from many linear theories of waves, where the nonlinear terms are often neglected as a first-order approximation.

Variance, Standard Deviation and Turbulence Intensity

One statistical measure of the dispersion of data about the mean is the variance, σ_A^2, defined by:

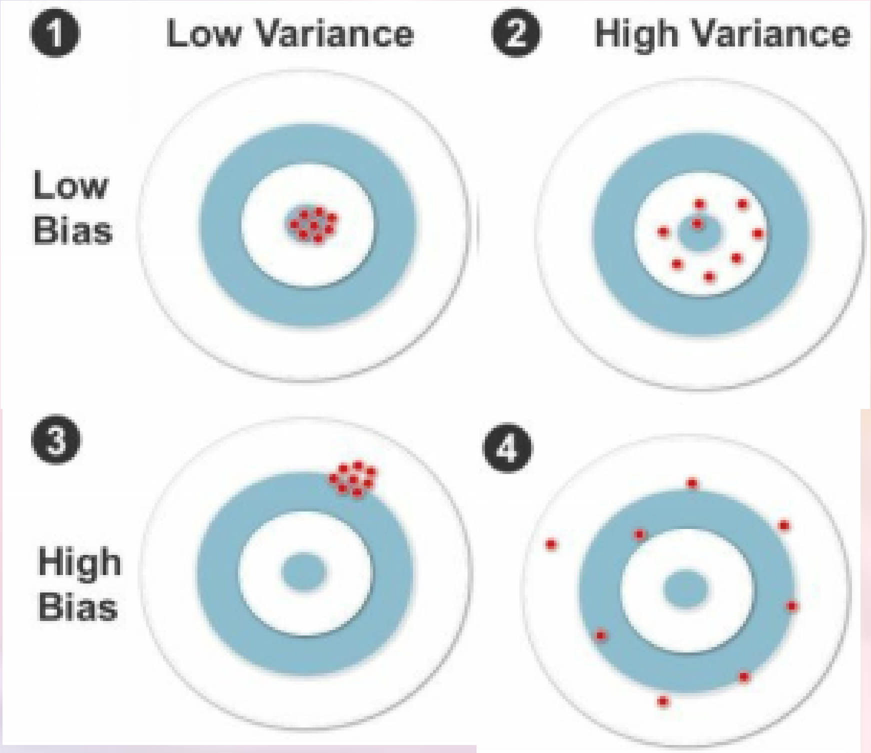
$$\sigma^2 = \frac{1}{N}\sum_{i=0}^{N-1}(A_i - \overline{A})^2$$

$$a' = A - \overline{A} \qquad\qquad \sigma^2 = \frac{1}{N}\sum_{i=0}^{N-1}(a_i')^2 = \overline{a'^2}$$

This is known as the biased variance.

It is a good measure of the dispersion of a sample of BL observations, but not the best measure of the dispersion of the whole population of possible observations. A better estimate of the variance (an unbiased variance) of the population, given a sample of data, is

$$\sigma_A^2 = \frac{1}{(N-1)}\sum_{i=0}^{N-1}(A_i - \overline{A})^2$$



❶
Low Variance
❷
High Variance
Low Bias
❸
High Bias
❹

Recall that the turbulent part (or the perturbation or gust part) of a turbulent variable is given by $A = \overline{A} + a'$

Substituting this into the biased definition of variance gives:

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} a'_i{}^2 = \overline{a'^2}$$

Thus, whenever we encounter the average of the square of a turbulent part of a variable, such as

$$\overline{u'^2}\ ,\ \overline{v'^2}\ ,\ \overline{w'^2}\ ,\ \overline{\theta'^2}\ ,\ \overline{r'^2}\ ,\ \text{or}\ \overline{q'^2}\ ,$$

we can interpret ese as variances.

The standard deviation is defined as the square root of the variance:

$$\sigma_A = \left(\overline{a'^2}\right)^{1/2}$$

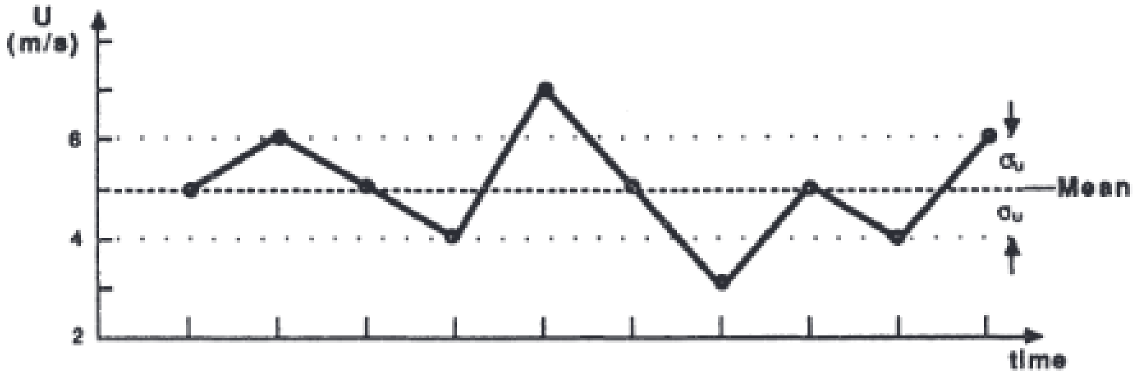


Fig. 2.6 Relationship between the standard deviation and turbulence variations. Solid line connects the data points; heavy dashed line is the average; dotted lines are drawn one standard deviation above and below the mean. Based on the example data in section 2.4.5.

$$I = \sigma_M / \overline{M}$$

Covariance and Correlation

In statistics, the covariance between two variables is defined as

$$\mathrm{covar}(A,B) \equiv \frac{1}{N}\sum_{i=0}^{N-1}(A_i - \overline{A})\cdot(B_i - \overline{B})$$

Using our Reynolds averaging methods, we can show that:

$$\mathrm{covar}(A,B) \equiv \frac{1}{N}\sum_{i=0}^{N-1} a_i'\, b_i' = \overline{a'\, b'}$$

The covariance indicates the degree of common relationship between the two variables, A and B.

$$r_{AB} \equiv \frac{\overline{a'\, b'}}{\sigma_A\, \sigma_B}$$

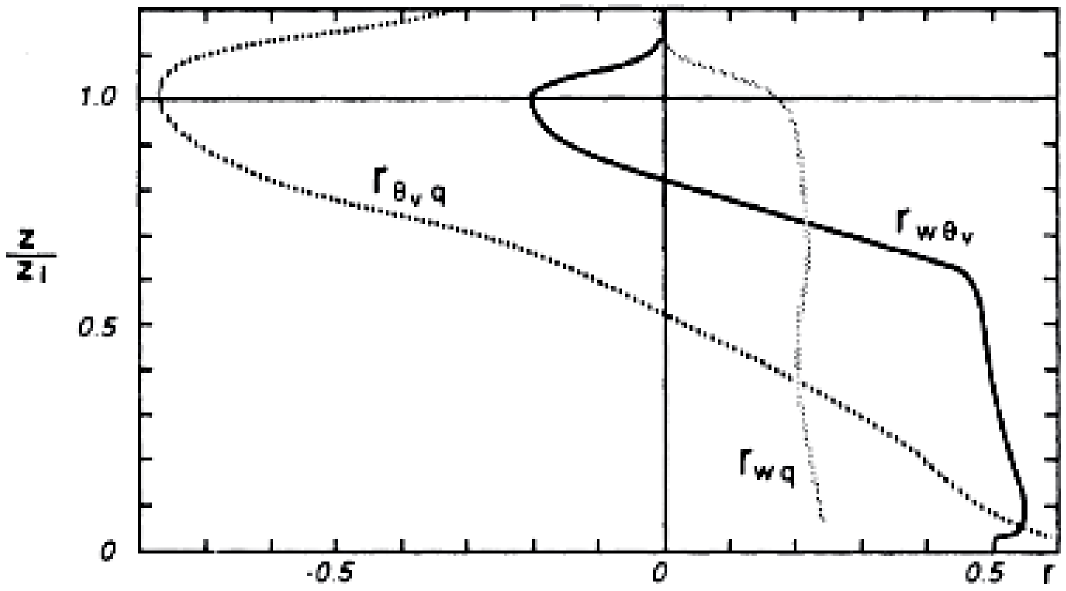
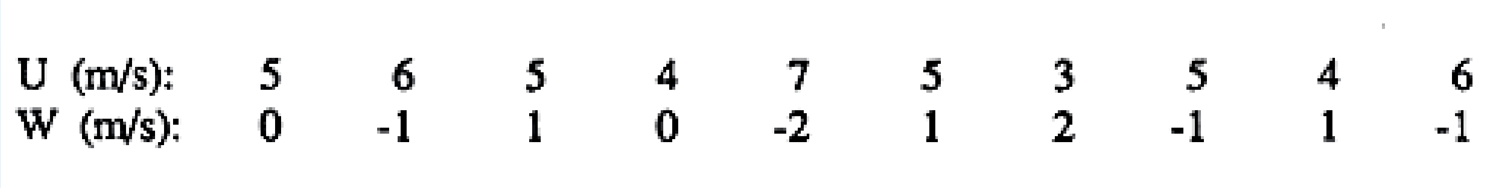


Fig. 2.7 Correlation coefficient profiles in the convective mixed layer.

Example

Suppose that we erect a short mast instrumented with anemometers to measure the U and W wind components. We record the instantaneous wind speeds every 6 s for a minute, resulting in the following 10 pairs of wind observations:



U (m/s):	5	6	5	4	7	5	3	5	4	6
W (m/s):	0	-1	1	0	-2	1	2	-1	1	-1

Find the mean, biased variance, and standard deviation for each wind component . Also, find the covariance and correlation coefficient between U and W.

$$\overline{U} = 5\ \mathrm{m \cdot s^{-1}}$$

$$\overline{W} = 0\ \mathrm{m \cdot s^{-1}}$$

$$\overline{u'w'} = -1.10\ \mathrm{m^2 \cdot s^{-2}}$$

$$\sigma_U^2 = 1.20\ \mathrm{m^2 \cdot s^{-2}}$$

$$\sigma_W^2 = 1.40\ \mathrm{m^2 \cdot s^{-2}}$$

$$\sigma_U = 1.10\ \mathrm{m \cdot s^{-1}}$$

$$\sigma_W = 1.18\ \mathrm{m \cdot s^{-1}}$$

$$r_{UW} = -0.85 \quad \text{(dimensionless)}$$

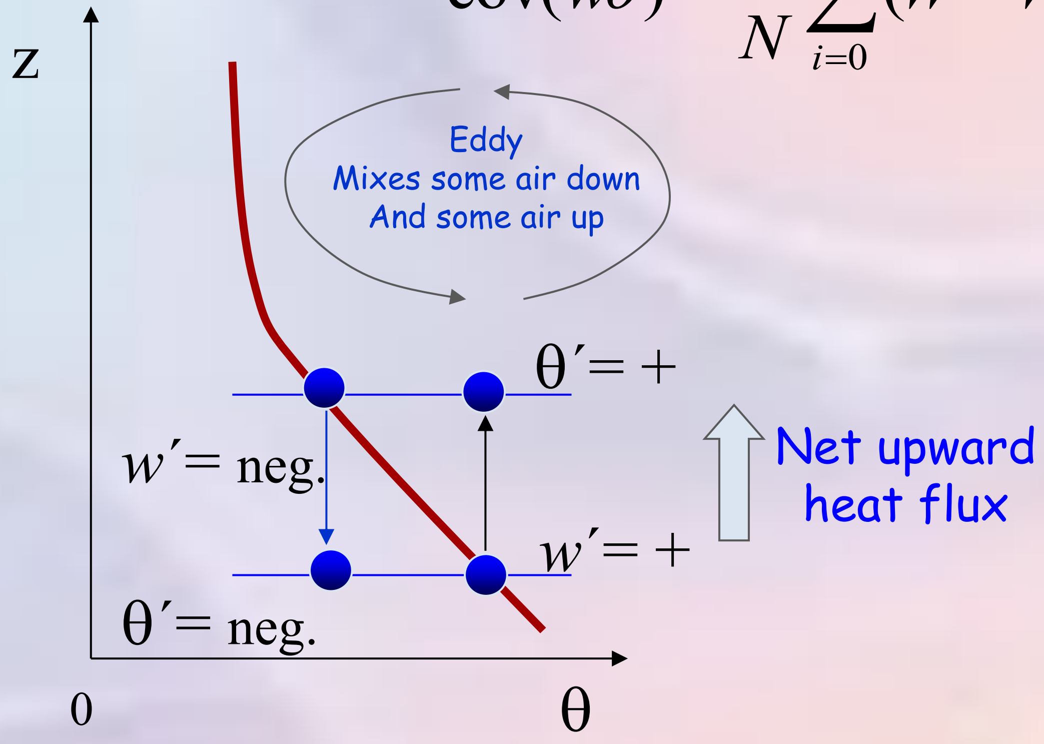
Turbulent Flux?

As a conceptual tool, suppose we examine a small idealized eddy near the ground on a hot summer day (see Fig.).

Transport of a quantity by eddies or swirls.

The covariance of a velocity component and any quantity.

$$\mathrm{cov}(w\theta) = \frac{1}{N}\sum_{i=0}^{N-1}(W-\overline{W})\cdot(\theta-\overline{\theta}) = \frac{1}{N}\sum_{i=1}^{N} w_i'\theta_i' = \overline{w'\theta'}$$



Turbulent Sensible Heat Flux

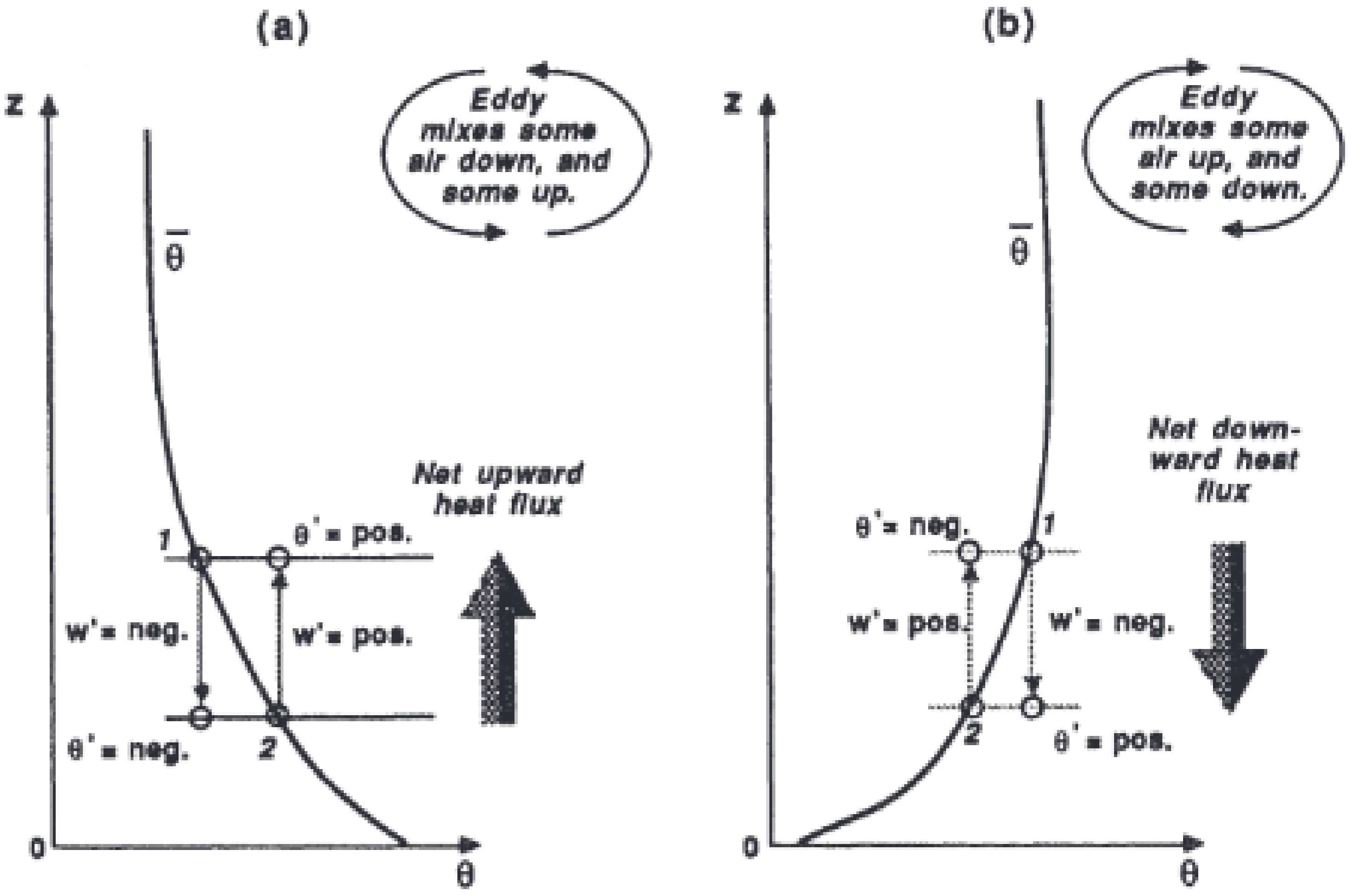


Fig. 2.12 Idealization of the small eddy mixing process, showing (a) net upward turbulent heat flux in a statically unstable environment, and (b) net downward turbulent heat flux in a stable environment.

As before, we can extend our arguments to write various kinds of eddy flux :

Vertical kinematic eddy heat flux $= \overline{w'\theta'}$

Vertical kinematic eddy moisture flux = $\overline{w'q'}$

x-direction kinematic eddy heat flux = $\overline{u'\theta'}$

Vertical kinematic eddy flux of u-momentum = $\overline{u'w'}$

The last flux is also the x-direction kinematic eddy flux of W-momentum.

Comparing the advective fluxes to the eddy fluxes, it is important to recognize that $\overline{W} \cong 0$ throughout most of the boundary layer.

Turbulence Kinetic Energy

The usual definition of kinetic energy (KE) is KE = 0.5 m M^2, where m is mass.

When dealing with a fluid such as air it is more convenient to talk about kinetic energy per unit mass, which is just 0.5 M^2

$$\mathrm{MKE/m} = \frac{1}{2}\left(\overline{U}^2 + \overline{V}^2 + \overline{W}^2\right)$$

$$e = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right)$$

where e represents an instantaneous turbulence kinetic energy per unit mass.

There is an additional portion of the total KE consisting of mean-turbulence products, but this disappears upon averaging.

By averaging over these instantaneous values, we can define a mean turbulence kinetic energy (TKE) that is more representative of the overall flow:

$$\frac{TKE}{m} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \bar{e}$$

We can immediately see the relationship between TKE/m and the definition of variance defined in the last section. It is apparent that statistics will play an important role in our quantification of turbulence.

The turbulence kinetic energy is one of the most important quantities used to study the turbulent BL.

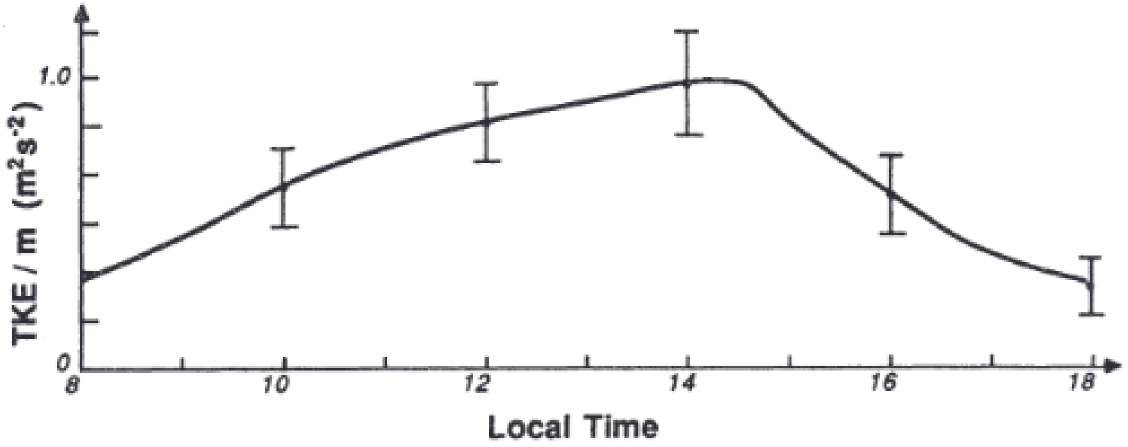


Fig. 2.8 Diurnal cycle of TKE/m = 1/2 ($\overline{u'^2}$+ $\overline{v'^2}$ +$\overline{w'^2}$) measured by aircraft below 300 m agl over Tennessee, August 1978. Adapted from Bean and Repoff, 1983.

A typical daytime variation of TKE in convective conditions is shown in Fig

Examples of the vertical profile of TKE for various boundary layers are shown in Fig.

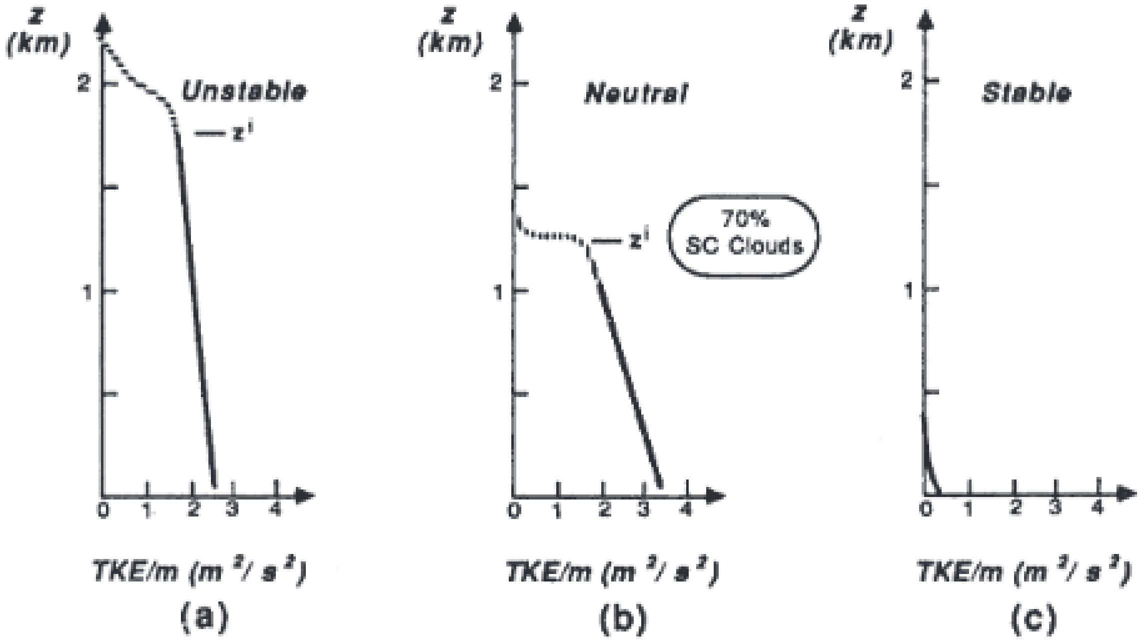


Fig. 2.9 Examples of turbulence kinetic energy per unit mass (TKE/m = $1/2\,(\overline{u'^2}+\overline{v'^2}+\overline{w'^2})$) for various boundary layers: (a) Daytime convective mixed layer with mostly clear skies and light winds 1600 local time, 14 June 1983, Chickasha, OK; (b) Near neutral day with strong winds (10-15 m/s at surface) and broken clouds at 1100 local time, 12 June 1983, Chickasha, OK; (c) Nocturnal stable boundary layer at 1000 local time, 11 September 1973 in Minnesota (from Caughey, et al, 1979,) Part (c) data excludes $\overline{v'^2}$ contributions because of measurement errors.